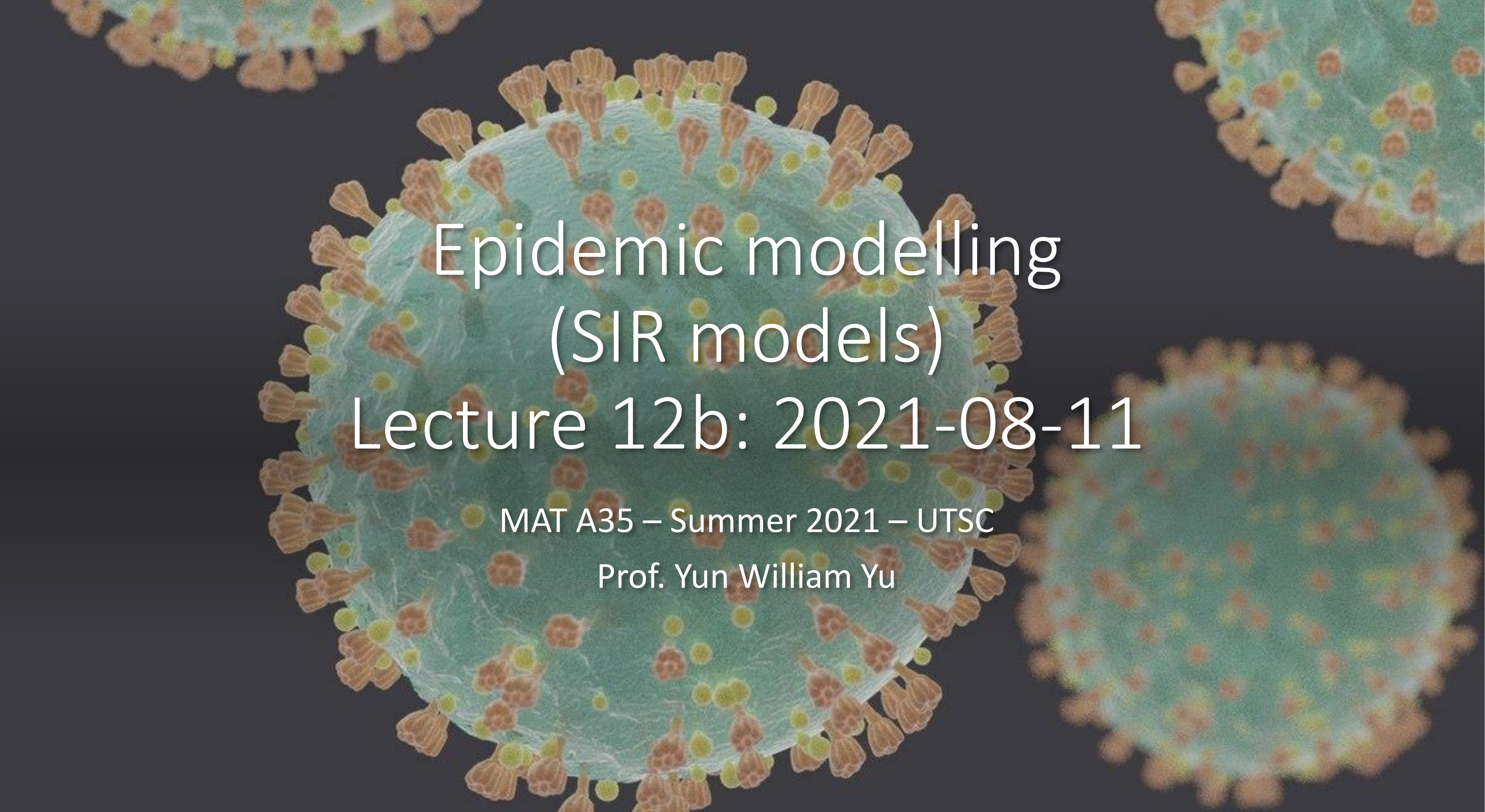


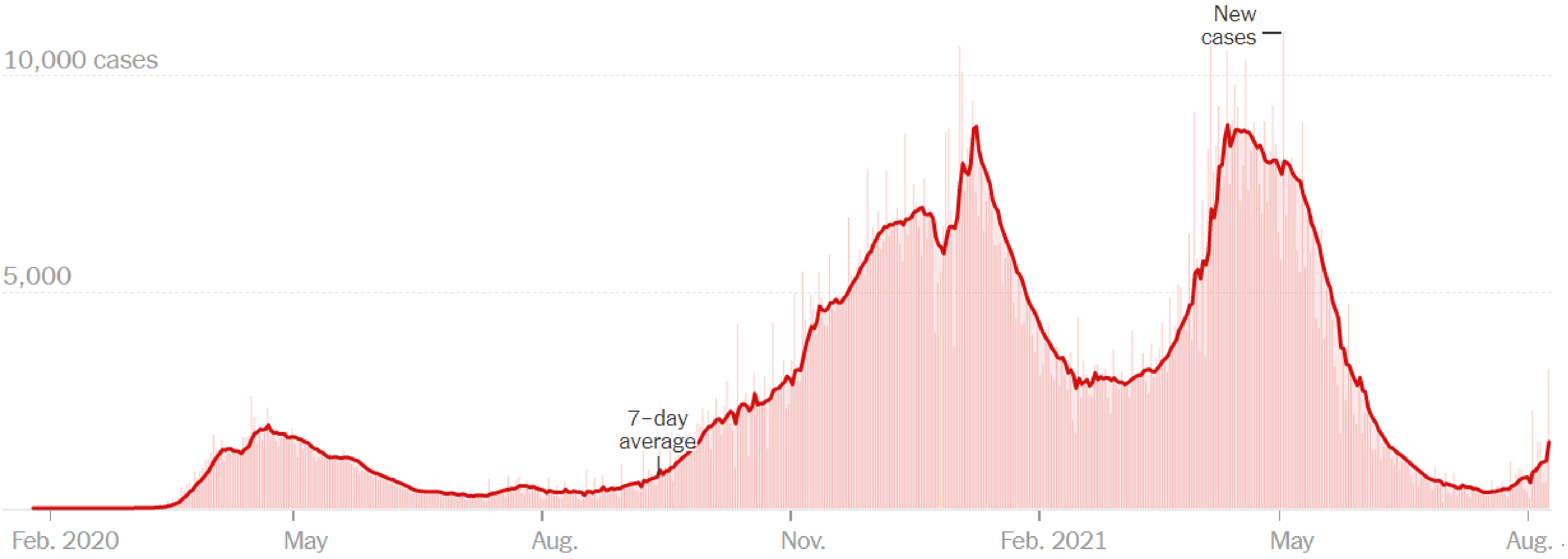
Epidemic modelling (SIR models) Lecture 12b: 2021-08-11

MAT A35 – Summer 2021 – UTSC

Prof. Yun William Yu

Image credit: https://commons.wikimedia.org/wiki/File:Coronavirus_SARS-CoV-2.jpg

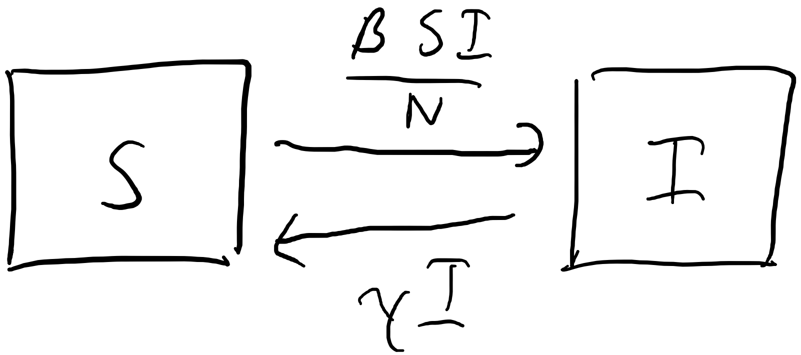
Epidemic curves – Covid19 in Canada



https://www.nytimes.com/interactive/2021/world/canada-covid-cases.html

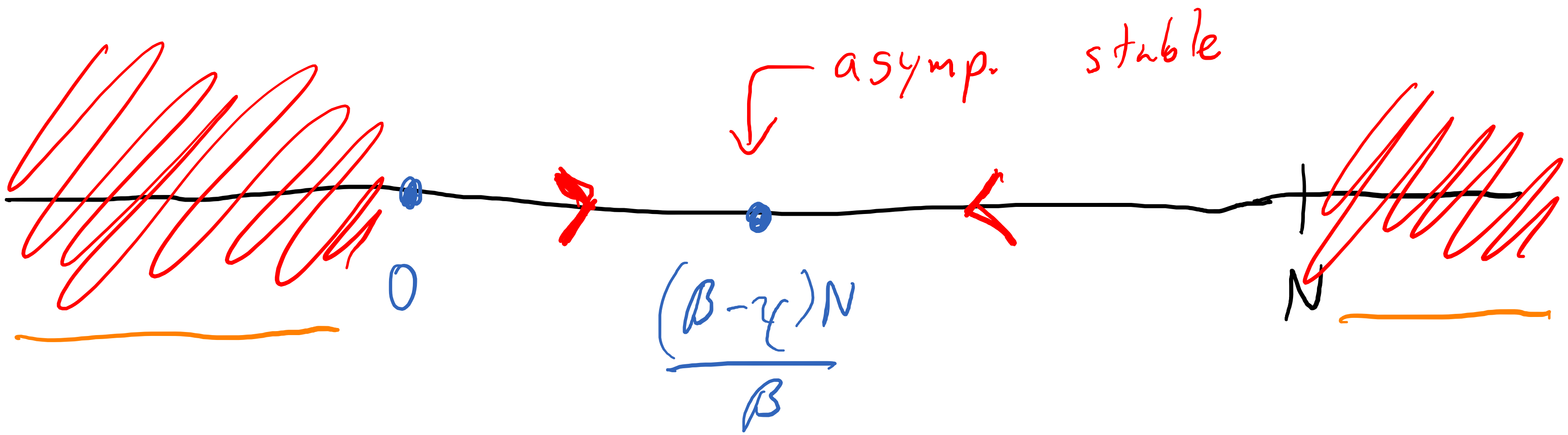
SIS Model

- Modified assumption 1: The infection rate is proportional to the average number of times an infected individual encounters a susceptible individual in the population, assuming random encounters.
- Assumption 2: there is a total fixed population size $N = I(t) + S(t)$, where S is the number of Susceptible individuals.
- Assumption 3: Individuals recover at rate γ, and become susceptible to re-infection.



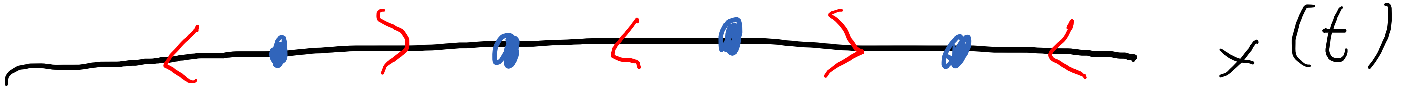
SIS Model Properties

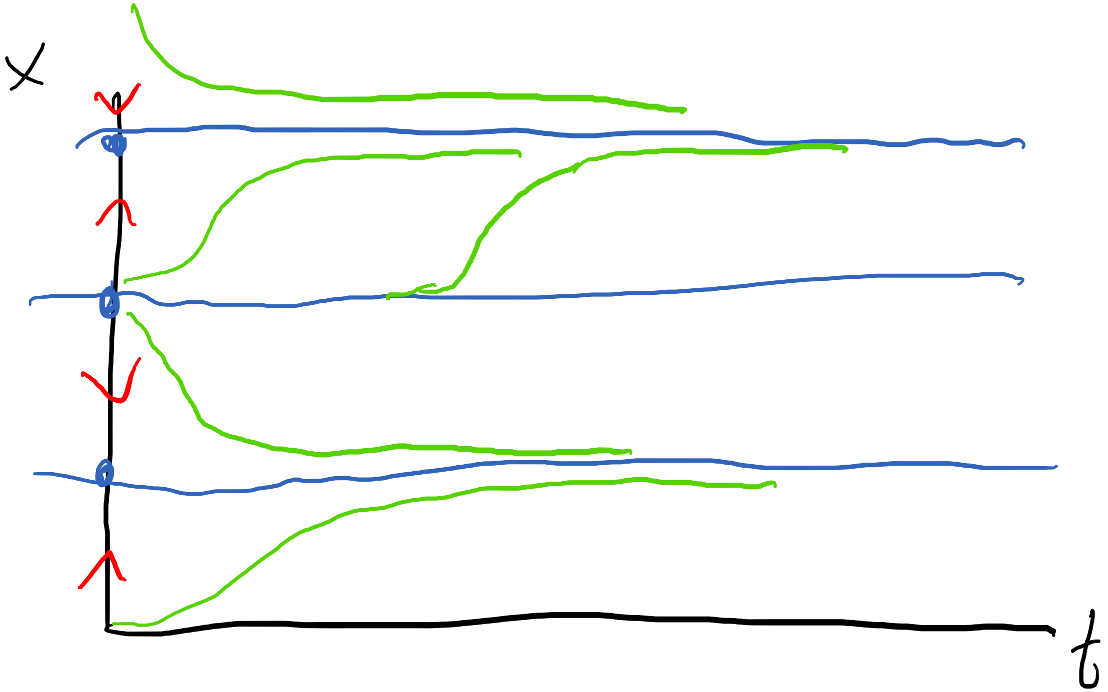
- In the SIS model, we call the ratio $R_0 = \frac{\beta}{\gamma}$ the *basic reproduction number* of the system because it is the number of secondary infections β caused during the infectious period $\frac{1}{\gamma}$.
- If $R_0 > 1$, the disease persists and we are at the endemic disease equilibrium.

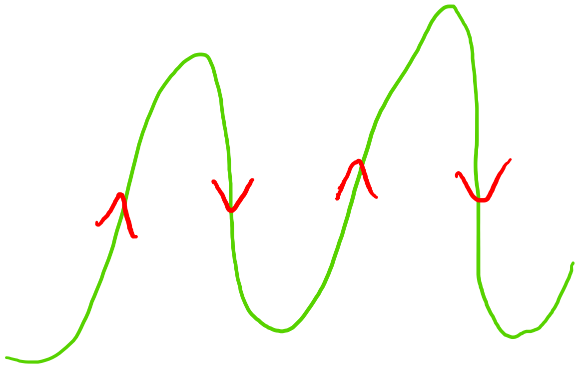


1D systems and complex behavior

- Autonomous 1D ODE systems $\dot{x} = f(x)$ can't have cycles or any doubling back because you'd have to cross a point going the other direction.







Modelling waves

- How can ODE models capture the infection waves that we see in real epidemics?

A: Add more variables
B: Make them nonautonomous
C: Add in births/deaths
D: All of the above
E: None of the above

- ~~Let’s try out a couple of options numerically~~.

SIR Model

- Modified assumption 3: Individuals recover at rate γ, and become *Removed* $R(t)$ from the population.
- Modified assumption 2: There is a total fixed population size $N = I(t) + S(t) + R(t)$.

$$\boxed{S} \xrightarrow{\frac{\beta SI}{N}} \boxed{I} \xrightarrow{\gamma I} \boxed{R}$$

$$\dot{S} = -\frac{\beta}{N} SI$$

$$\dot{I} = \frac{\beta}{N} SI - \gamma I$$

$$\dot{R} = \gamma I$$

and $N = S + I + R$

Reduce SIR model to two variables

- $\begin{cases} \dot{S} = -\frac{\beta}{N} SI \\ \dot{I} = \frac{\beta}{N} SI - \gamma I \\ \dot{R} = \gamma I \end{cases}$

$N = S + I + R$

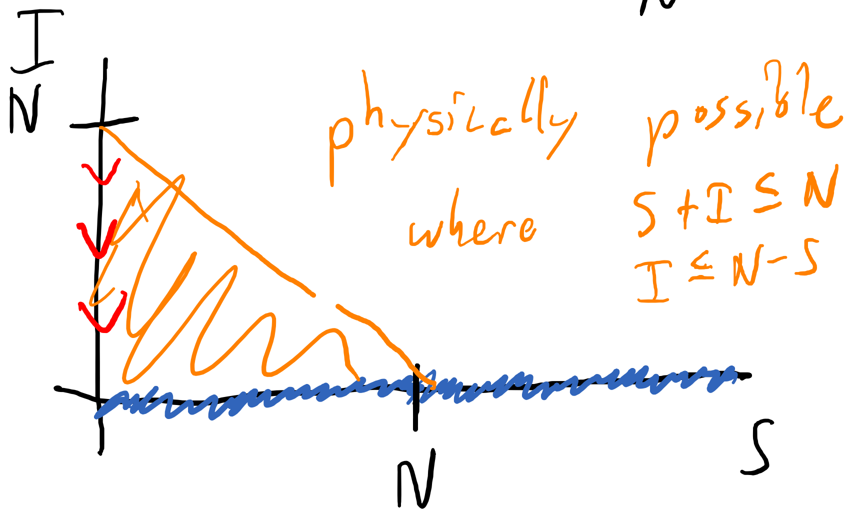
$\Rightarrow \quad R = N - S - I$

$\begin{cases} \dot{S} = -\frac{\beta}{N} SI \\ \dot{I} = \frac{\beta}{N} SI - \gamma I \end{cases}$

Equilibria:

$\left.\begin{aligned} 0 &= -\frac{\beta}{N} SI \\ 0 &= \frac{\beta}{N} SI - \gamma I \end{aligned}\right\}$ $\quad \gamma I = 0$, $\quad I = 0$

S can be anything



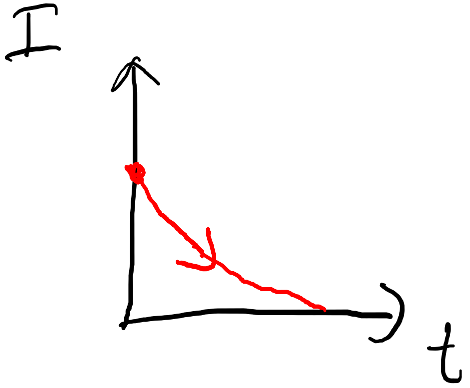
If $S = 0$, $\dot{I} < 0$.

Plots of basic SIR model

infection rate

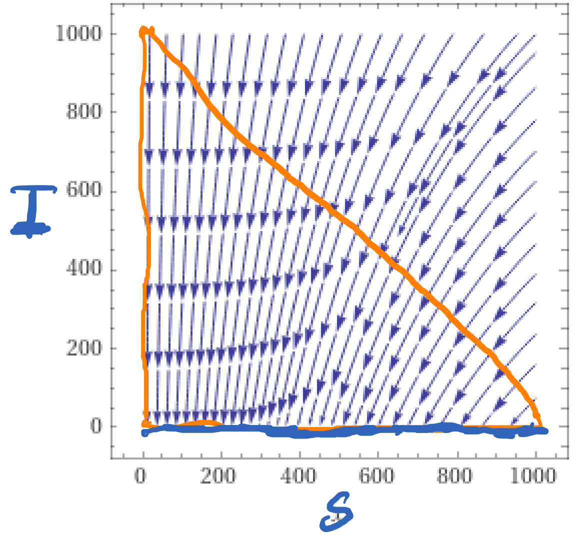
recovery rate

- Let $\beta = 0.1, \gamma = 0.2, N = 1000$
- $\begin{cases} \dot{S} = -\frac{0.1}{1000} SI \\ \dot{I} = \frac{0.1}{1000} SI - 0.2I \end{cases}$

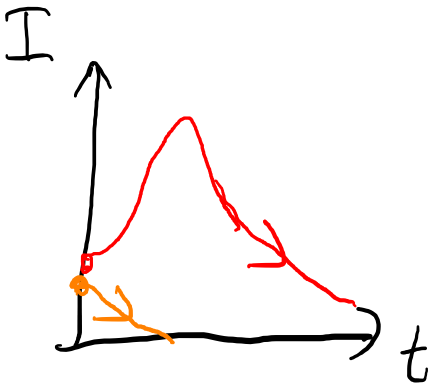


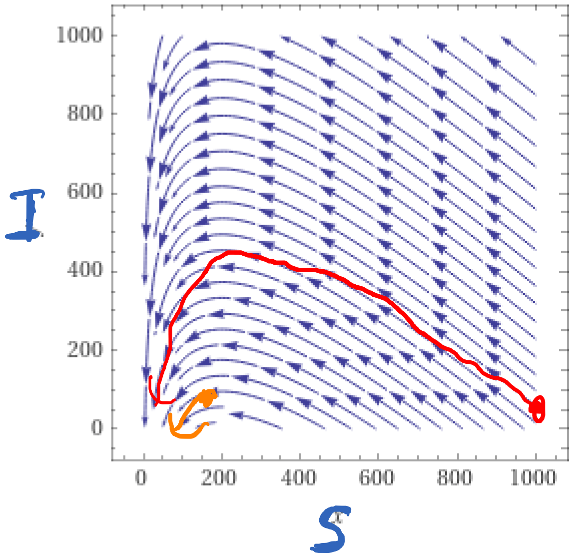
WolframAlpha: streamplot {-1/10000 * x * y, x*y/10000 - 0.2*y}, x=0..1000, y=0..1000

S + I + R = N



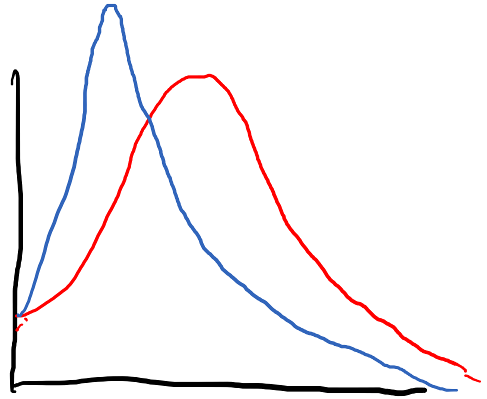
- Let $\beta = 0.1, \gamma = 0.02, N = 1000$
- $\begin{cases} \dot{S} = -\frac{0.1}{1000} SI \\ \dot{I} = \frac{0.1}{1000} SI - 0.02I \end{cases}$





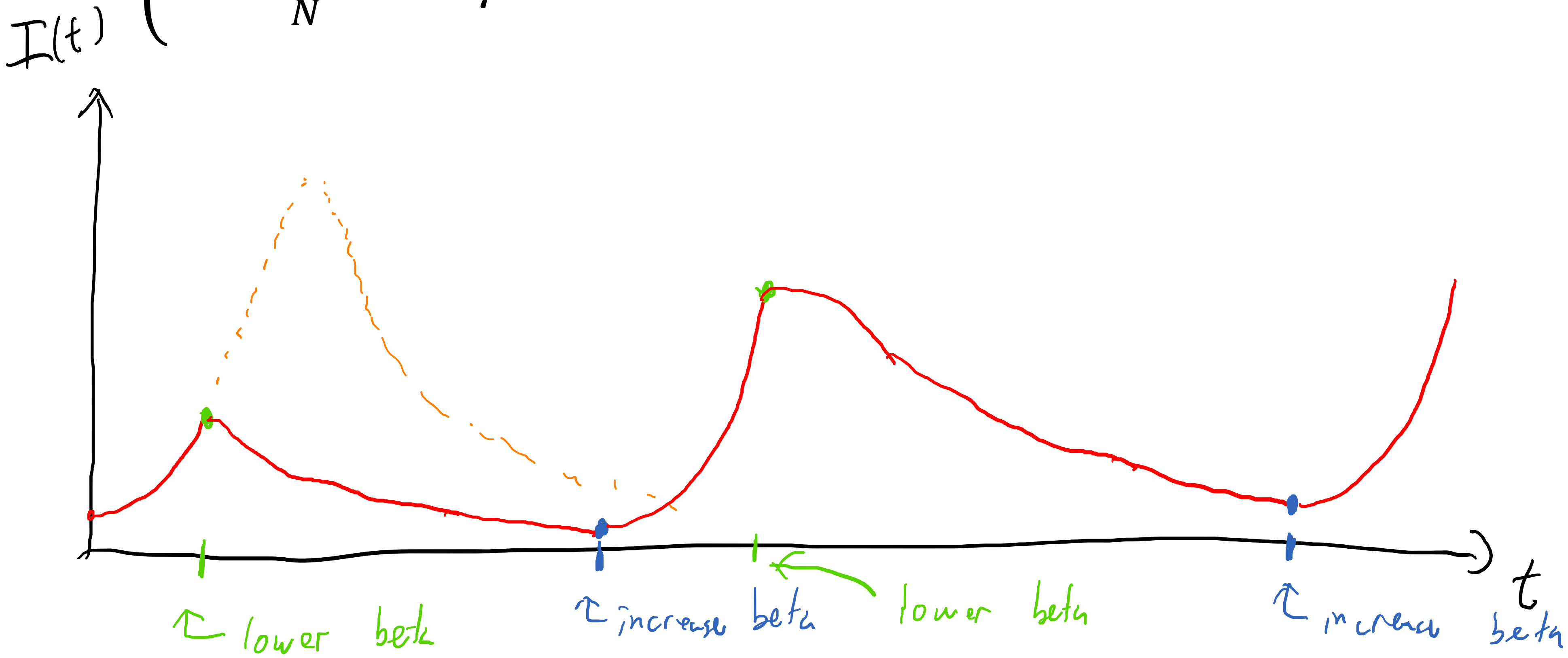
Basic SIR model gives a single wave

- Again, $R_0 = \frac{\beta}{\gamma}$.
- If $R_0 < 0$, then no epidemic happens because infected individuals never increase.
- If $R_0 > 1$, then an epidemic with a single wave happens.
- Then higher the R_0, the higher the peak. We can "flatten the curve" by reducing β, the transmission rate.
- https://www.geogebra.org/m/p5ucts38 by Ivan De Winne



Nonautonomous Epidemic model

- What if β is sometimes high, and sometimes low, as a function of time?
- Then $\begin{cases} \dot{S} = -\dfrac{\beta(t)}{N} SI \\ \dot{I} = \dfrac{\beta(t)}{N} SI - \gamma I \end{cases}$ is not autonomous.



(autonomous) SIRS Epidemic Model

nu

- Assumption 4: Removed individuals lose immunity at rate ν.

$$
\begin{tikzcd}
\boxed{S} \arrow[r, "\frac{\beta SI}{N}"] & \boxed{I} \arrow[r, "\gamma I"] & \boxed{R} \arrow[ll, bend left, "\nu R"]
\end{tikzcd}
$$

$$
\left.\begin{aligned}
\dot{S} &= -\frac{\beta}{N} SI + \nu R \\
\dot{I} &= \frac{\beta}{N} SI - \gamma I \\
\dot{R} &= \gamma I - \nu R
\end{aligned}\right\}
$$

$$N = S + I + R$$

$$R = N - S - I$$

$$\dot{S} = -\frac{\beta}{N} SI + \nu (N - S - I)$$

$$\dot{I} = \frac{\beta}{N} SI - \gamma I$$

Phase plane analysis of SIRS model

- $$\begin{cases} \dot{S} = -\frac{\beta}{N}SI + \nu(N - I - S) \\ \dot{I} = \frac{\beta}{N}SI - \gamma I \end{cases}$$

Equilibria:

- $0 = -\frac{\beta}{N}SI + \nu(N-I-S)$

$0 = \frac{\beta}{N}SI - \gamma I = I\left(\frac{\beta}{N}S - \gamma\right) \Rightarrow I = 0$ or $S = \frac{N\gamma}{\beta}$

Case 1: $I = 0$

$\Rightarrow 0 = \nu(N-S)$

$\Rightarrow S = N$

Case 2: $S = \frac{N\gamma}{\beta}$

$0 = -\frac{\beta}{N} \cdot \frac{N\gamma}{\beta} \cdot I + \nu\left(N - I - \frac{N\gamma}{\beta}\right)$

$\gamma I + \nu I = \nu N\left(1 - \frac{\gamma}{\beta}\right) \Rightarrow I = \frac{\nu N\left(1 - \frac{\gamma}{\beta}\right)}{\gamma + \nu}$

Qualitative analysis of equilibria

- $$\begin{cases} \dot{S} = -\frac{\beta}{N} SI + \nu(N - I - S) \\ \dot{I} = \frac{\beta}{N} SI - \gamma I \end{cases}$$

$$J(S,I) = \begin{bmatrix} -\frac{\beta I}{N} - \nu & -\frac{\beta S}{N} - \nu \\ \frac{\beta I}{N} & \frac{\beta S}{N} - \gamma \end{bmatrix}$$

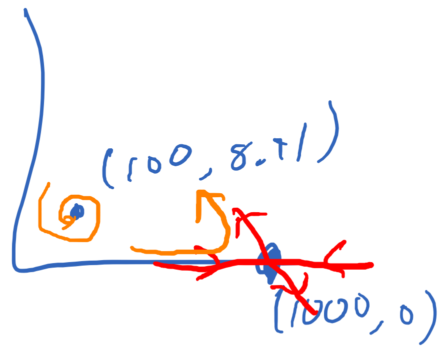
- Equilibria at $\partial_S \quad \partial_I$

$$\begin{cases} (S,I) = (N,0) \\ (S,I) = \left(\frac{N\gamma}{\beta}, \frac{\nu N\left(1-\frac{\gamma}{\beta}\right)}{\gamma+\nu}\right) \end{cases}$$

$$J(N,0) = \begin{bmatrix} & -\beta-\nu \\ \beta & \beta-\gamma \end{bmatrix}$$

$$J(\ldots,\ldots) = \begin{bmatrix} -\frac{\nu\beta\left(1-\frac{\gamma}{\beta}\right)}{\gamma+\nu} - \nu & -\gamma-\nu \\ \frac{\nu\beta\left(1-\frac{\gamma}{\beta}\right)}{\gamma+\nu} & 0 \end{bmatrix}$$

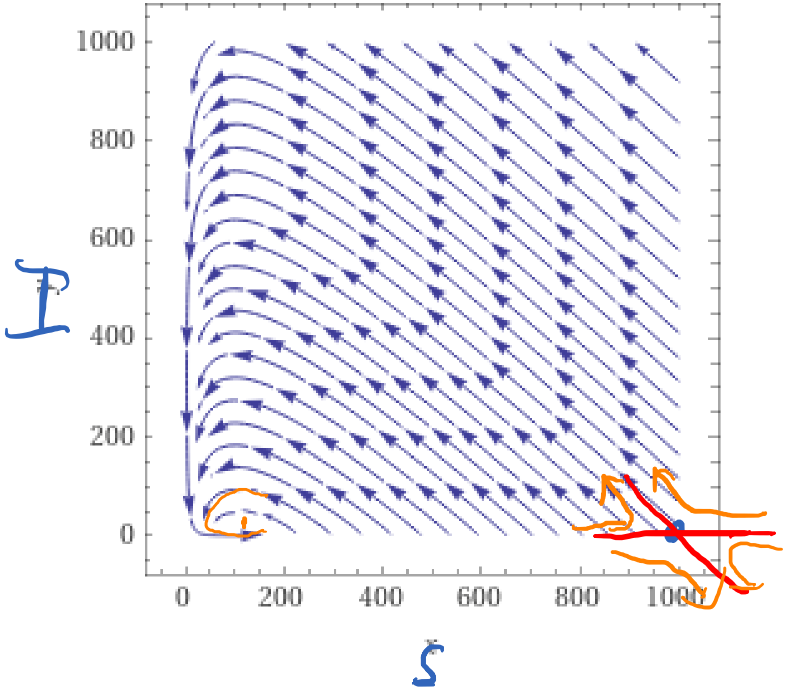
Example with periodic behavior



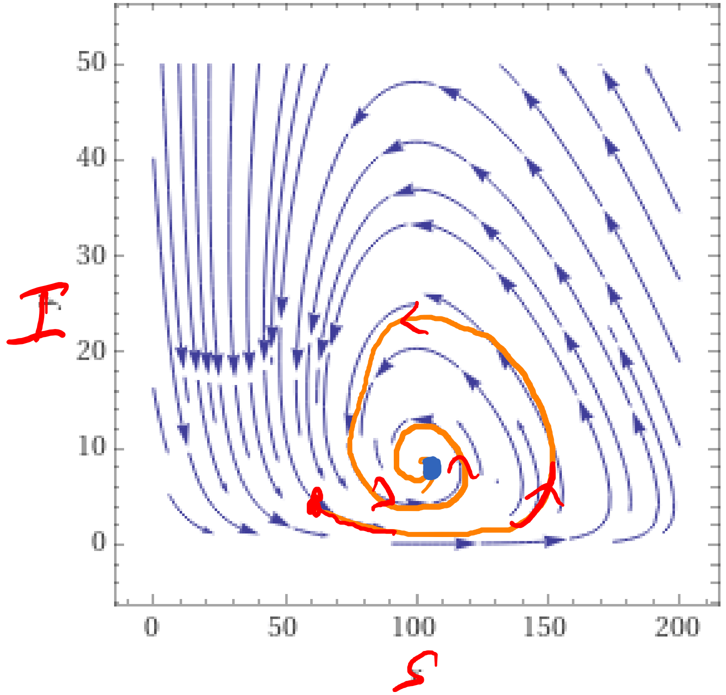
- $\beta = 0.1, \gamma = 0.01, \nu = 0.0001, N = 1000$
- $\dot{S} = -\frac{\beta}{N} SI + \nu(N - I - S) = -0.0001SI + 0.0001(1000 - S - I)$
- $\dot{I} = \frac{\beta}{N} SI - \gamma I = 0.0001SI - 0.01I$
- Equilibria: $(S, I) = (1000,0)$ or $(S, I) \approx (100, 8.91)$
- Jacobian $J(S, I) = \begin{bmatrix} -0.0001I - 0.0001 & -0.0001S - 0.0001 \\ 0.0001I & 0.0001S - 0.01 \end{bmatrix}$
- $J(1000,0) = \begin{bmatrix} -0.0001 & -0.1001 \\ 0 & 0.09 \end{bmatrix}$, $\lambda_1 = -0.0001, \lambda_2 = 0.09$, saddle pt
- $J(100, 8.91) = \begin{bmatrix} -0.000991 & -0.0101 \\ 0.000891 & 0 \end{bmatrix}$, $\lambda_{1,2} \approx -0.0004955 \pm 0.00296i$
  so the equilibrium is an attracting spiral

Visualization

- streamplot {- x * y/10000 + 0.0001*(1000-x-y), x*y/10000 - 0.01*y}, x=0..1000, y=0..1000

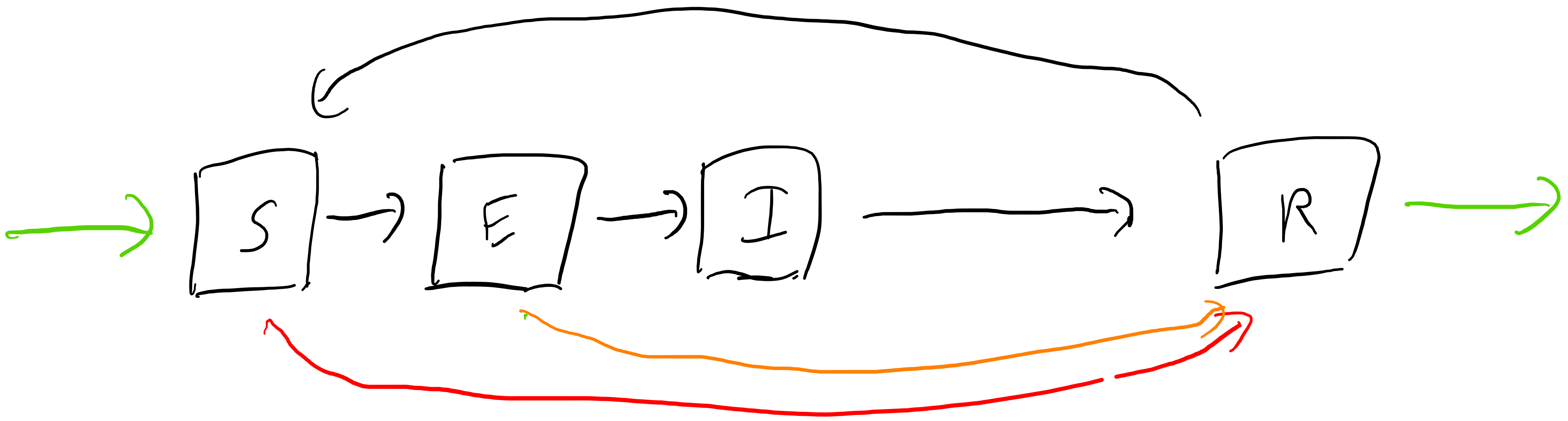


- streamplot {- x * y/10000 + 0.0001*(1000-x-y), x*y/10000 - 0.01*y}, x=0..400, y=0..50

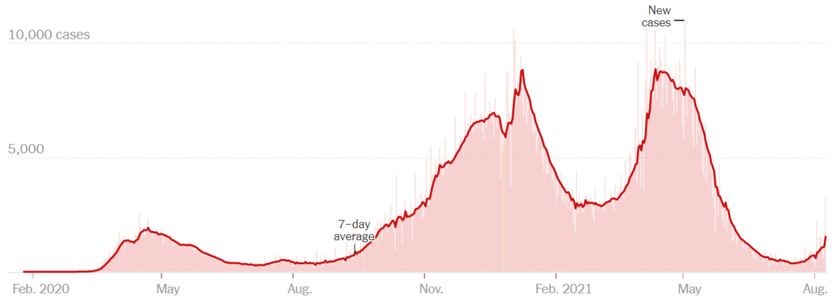


More sophisticated epidemic models

- Vaccinations = arrow directly from S → R
- Birth/Death = arrows entering S or leaving R
- Exposed but not Infectious = new compartment



Modelling summary



- We can build models of complex biological phenomenon like epidemics by looking at behavior we care about and adding more assumptions to reproduce that real behavior in our models.
- Sometimes, we can get similar behavior with two different assumptions, so as mathematical biologists we need to decide what fits better.
- For epidemic modelling, these can include additional variables/compartments, as well as new arrows between compartments.