Week 1: Integration
Lecture 2 – 2021-05-12

MAT A35 – Summer 2021 – UTSC

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Basic differentiation rules

Derivative rule	Integration rule
$\frac{d}{dx}[kx] = k$	$\int k\,dx = kx + C$
$\frac{d}{dx}\left[\frac{x^{r+1}}{r+1}\right] = x^r, \qquad r \neq -1$	$\int x^r\,dx = \frac{x^{r+1}}{r+1} + C, \qquad r \neq -1$
$\frac{d}{dx}[\ln\lvert x\rvert] = \frac{1}{\mathrm{x}} = \mathrm{x}^{-1}$	$\int x^{-1}\,dx = \ln\lvert x\rvert + C$
$\frac{d}{dx}\left[\frac{1}{a}e^{ax}\right] = e^{ax}$	$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$
$\frac{d}{dx}\left[-\frac{1}{a}\cos ax\right] = \sin ax$	$\int \sin ax\ \,dx = -\frac{1}{a}\cos ax + C$
$\frac{d}{dx}\left[\frac{1}{\mathrm{a}}\sin ax\right] = \cos ax$	$\int \cos ax\ \,dx = \frac{1}{\mathrm{a}}\sin ax + C$
$\frac{d}{dx}\left[\frac{1}{\mathrm{a}}\tan ax\right] = \sec^2 ax$	$\int \sec^2 ax\ \,dx = \frac{1}{\mathrm{a}}\tan ax + C$
$\frac{d}{dx}\left[-\frac{1}{\mathrm{a}}\cot ax\right] = \csc^2 ax$	$\int \csc^2 ax\ \,dx = -\frac{1}{\mathrm{a}}\cot ax + C$
$\frac{d}{dx}\left[\frac{1}{a}\sec ax\right] = \sec ax \tan ax$	$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec x + C$
$\frac{d}{dx}\left[-\frac{1}{a}\csc ax\right] = \csc ax \cot ax$	$\int \csc ax \cot ax\,dx = -\frac{1}{\mathrm{a}}\csc ax + C$

Area under curve

Riemann sums and trapezoid rule

- We can approximate area under any curve by dividing into shapes we know how to compute area for, like rectangles or trapezoids

Example

- Approximate the area under the parabola $y = x^2$ between 0 and 3 using a Riemann sum with 3 rectangles.

Try it out

- Approximate the area under the line $y = x$ between 0 and 4 using a Riemann sum.

More rectangles

- Another way to decrease approximation error is to use more rectangles.

Infinite rectangles!

- Take the limit as the rectangles become infinitely thin.

Definition: Let f be a continuous function on $[a, b]$ with $a < b$. Then the definite integral of f from a to b is defined by

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_i)$$

where $\Delta x = \frac{1}{n}(b - a)$ and $x_i = a + i\Delta x$. a and b are the *limits of integration*. If $f(x) > 0$ on $[a, b]$, then the definite integral represents the area between the curve $y = f(x)$ and the x-axis.

Riemann sum example: $\int_0^4 x^2 dx$

Signed Area

The definite integral gives a signed area, which is positive when the function is positive and negative when the function is negative.

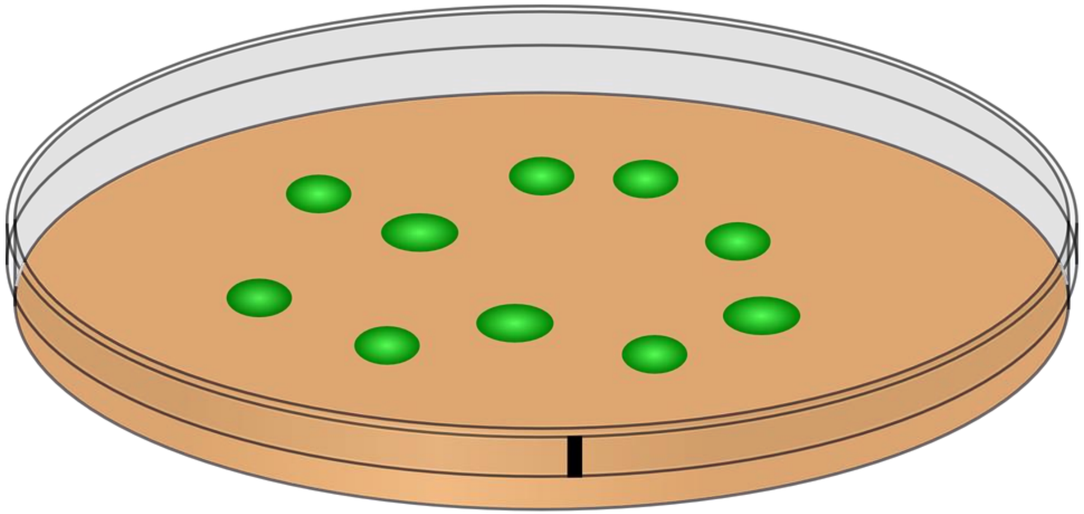
Fundamental Theorem of Calculus

- First form of the Fundamental Theorem of Calculus
  - Let f be a continuous function and let $A(x) = \int_a^x f(t)dt$. Then $A'(x) = f(x)$
  - If you integrate a function and then take the derivative, you get the same function back.
- Second form of the Fundamental Theorem of Calculus
  - Let $f(x)$ be a continuous function and suppose that $g'(x) = f(x)$ (i.e. $g(x)$ is an antiderivative of $f(x)$). Then $\int_a^b f(x)dx = g(b) - g(a)$
  - You can use the antiderivative of a function to compute the definite integral without explicitly using infinite Riemann sums.

Example

Application

- Bacteria in a petri dish grow at a rate of $P'(t) = 100e^{-t}$ cells per hour, where t is time in hours. Determine how much the population increases from time $t = 0$ to time $t = 2$.



Application

- Corn needs 1.5 inches of rainfall or watering per week.
- Suppose it rains today between noon and 1pm at a rate of $f(t) = 2 - t^2$ inches/hour, where t is the number of hours since noon.
- Did it rain enough that you do not need to water your corn field?

Average of a function

- Let $f\colon [\mathrm{a},\mathrm{b}] \to \mathbb{R}$ be a continuous function. Then its average value $y_{av} = \frac{1}{b-a}\int_a^b f(x)dx$.

Properties of definite integrals

- Constant multiplication: $\int_a^b k \cdot f(x)dx = k \cdot \int_a^b f(x)dx$

- Sum of different integrands with same bounds
  - $\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

- Sum of same integrand with touching bounds
  - $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$ where $a < b < c$

Try it out

$$\int_0^2 x^2 dx + \int_0^2 5 dx + \int_1^3 (x^2+5) dx - \int_1^2 (x^2+5) dx$$

Area between curves

Let f and g be continuous functions, and suppose that $f(x) \geq g(x)$ over the interval $[a, b]$. Then the area of the region between the two curves on that interval is $\int_a^b [f(x) - g(x)]dx$.

When $[a, b]$ are unknown, can compute the intersection points to figure out the area bounded by curves.

Example

- Find the area bounded by the graphs of $f(x) = 2x - 2$ and $g(x) = x^2 - 2$.

Try it out

- Find the area bounded by graphs of $f(x) = x^2$ and $g(x) = x$.
- Step 1: find the intersection points.

- Step 2: Decide which graph is on top.

- Step 3: Compute the integral.

Chain rule → Substitution rule

- Chain rule: Let $f = f(u)$ be a function of u and $u = u(x)$ be a function of x. Then $\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$.

- "u-substitution" is the opposite of the chain rule.

Substitution rule algorithm

- Step 1: Guess an appropriate u
- Step 2: Compute du, dx, and x
- Step 3: Substitute in to get rid of all the x's
- Step 4: Integrate as a function of u
- Step 5: Convert back to x's

Example

Substitution for definite integrals

Try it out

- $\int_0^2 \frac{x}{(1+x^2)^2}\,dx$

- $\int \tan x\,dx$. Hint: $\tan x = \frac{\sin x}{\cos x}$. Let $u = \cos x$

Integration techniques – partial fractions

- Sometimes, it is easier to integrate if you break up a complicated expression into several simpler ones. One way to do this is with a partial fractions decomposition:

$$\frac{h(x)}{f(x)g(x)} = \frac{A(x)}{f(x)} + \frac{B(x)}{g(x)}$$

Where $h(x), f(x), g(x), A(x), B(x)$ are all polynomials in x.

Example

Try it out: $\int \frac{5x+1}{2x^2-x-1}\,dx$

Product Rule → Integration by parts

- Recall $\frac{d}{dx}[u(x)v(x)] = u(x)v'(x) + u'(x)v(x)$

- Integration by parts is the opposite of the product rule:
  - $\frac{d}{dx}[u(x)v(x)] = u(x)v'(x) + u'(x)v(x) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$
  - $d[u(x)v(x)] = u \cdot dv + v \cdot du$
  - $u \cdot dv = d[u(x)v(x)] - v \cdot du$
  - $\int u \cdot dv = \int d[u(x)v(x)] - \int v \cdot du$
  - $\int u\, dv = uv - \int v\, du$

Integration by parts algorithm

- $\int u\, dv = uv - \int v\, du$
- Step 1: Guess which part is u and which part is dv
- Step 2: Apply the formula above and hope you can solve $\int v\, du$
- Step 3: If it doesn't, try again with a different guess for u and dv.

- Step ?: Give up if no guess seems to work. The integral might not be amenable to integration by parts.

Example ($\int u\, dv = uv - \int v\, du$)

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Try it out: $\int x^2 e^x dx$