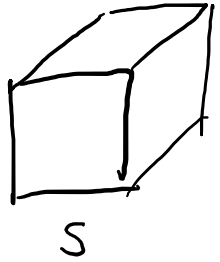
Volume and improper integration
Lecture 2b – 2021-05-19

MAT A35 – Summer 2021 – UTSC

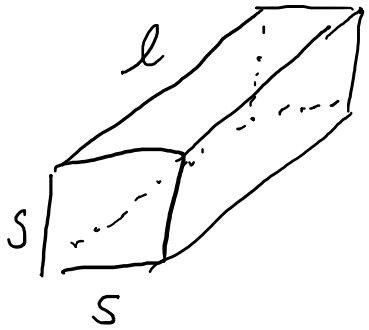
Prof. Yun William Yu

Volume of simple solids



cube

$Vol = s^3$



rectangular block

$Vol = s^2 \ell$

Area length

cylinder

$Vol = \pi r^2 h$

Area x height

Vase ?

Invention of pottery?

A: Before 1 AD

B: 1-1000 AD

C: 1000-1500 AD

D: 1500-1800 AD

E: 1800 AD-present

Solids of revolution

- Area under a curve can be approximated by rectangles

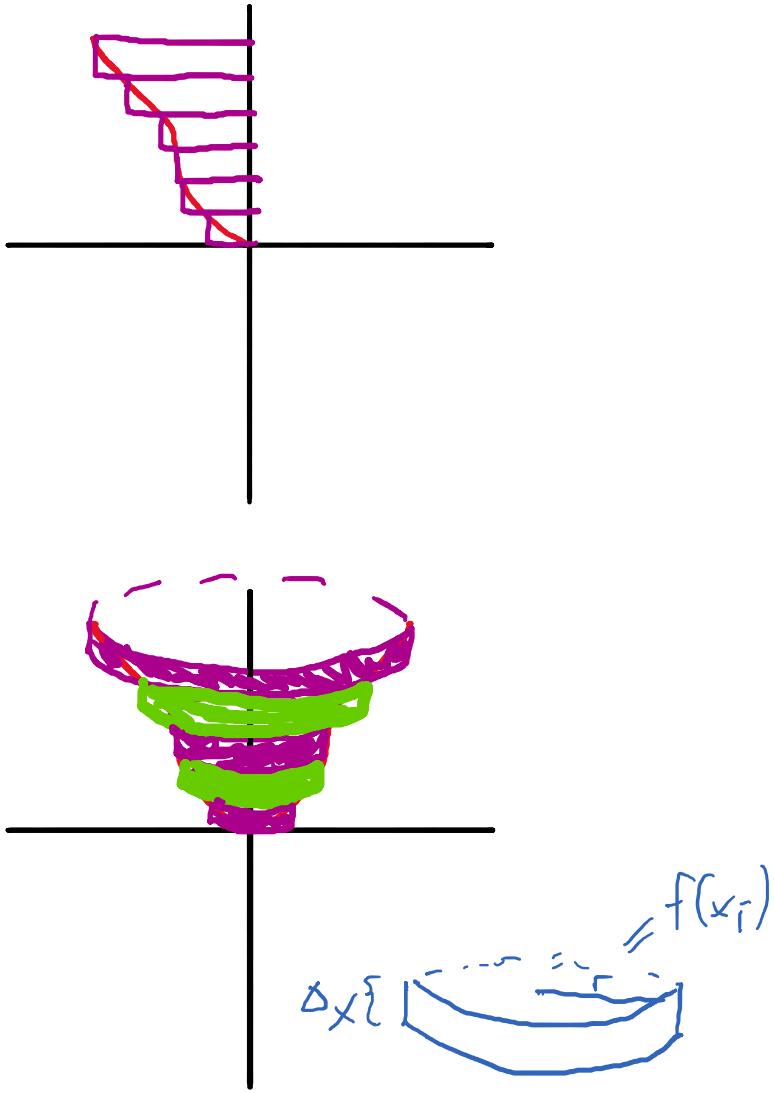
$$A = \lim_{n\to\infty} \sum_{1}^{n} f(x_i)\Delta x$$

$$= \int_a^b f(x)\,dx$$

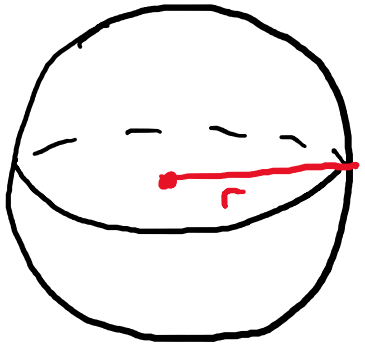
- What if we rotate about the vertical axis? What is the volume?

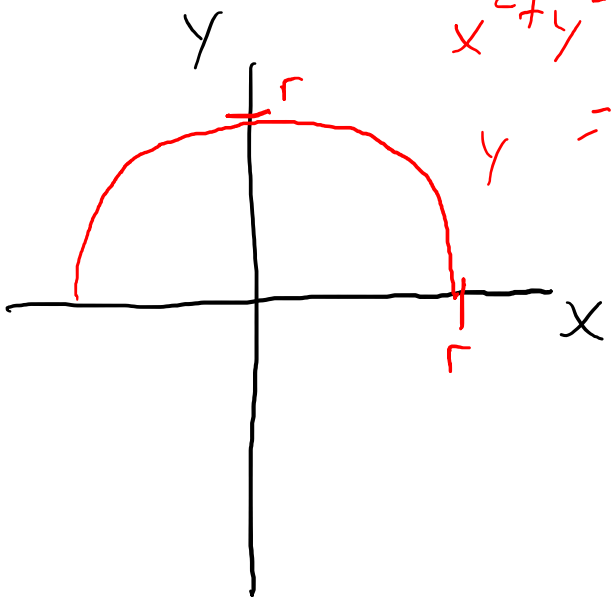
$$V = \lim_{n\to\infty} \sum_{1}^{n} \pi\big(f(x_i)\big)^2 \Delta x$$

$$= \int_a^b \pi\,(f(x))^2\,dx$$



Example – Volume of a sphere





$x^2+y^2=r^2$

$y=\sqrt{r^2-x^2}=f(x)$ $\sqrt{r^2-x^2}$

$$\int_{-r}^{r}\pi\left(f(x)\right)^2\,dx \quad //$$

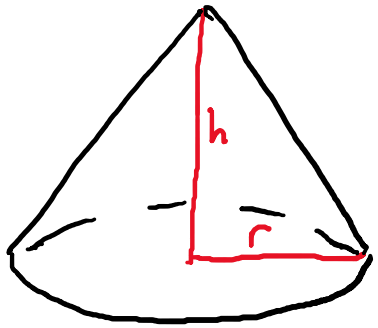
$$=2\pi\int_{0}^{r}\left(r^2-x^2\right)dx$$

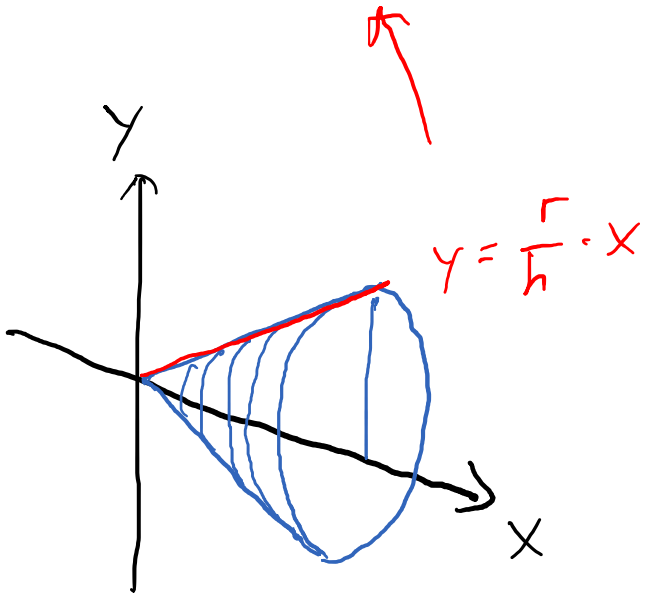
$$=2\pi\left[r^2x-\frac{1}{3}x^3\right]\Bigg|_0^r$$

$$=2\pi\left[r^3-\frac{1}{3}r^3\right]=2\pi\cdot\frac{2}{3}\cdot r^3$$

$$=\boxed{\frac{4}{3}\pi r^3}$$

Example – Volume of cone





$$\int_0^h \pi \left(\frac{r}{h} \cdot x \right)^2 dx$$

$$= \frac{\pi r^2}{h^2} \int_0^h x^2 \, dx$$

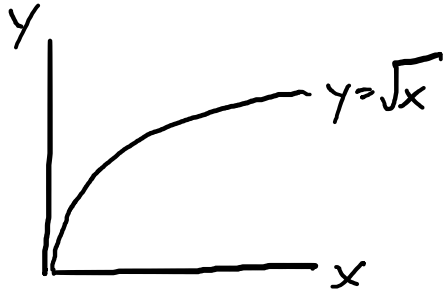
$$= \frac{\pi r^2}{h^2} \left[\frac{1}{3} x^3 \right] \Big|_0^h$$

$$= \frac{1}{3} \cdot \pi r^2 h$$

Try it out

$$Vol = \int_a^b \pi (f(x))^2 \, dx$$

- Find the volume of the solid of revolution generated by rotating the region under the graph of $y = \sqrt{x}$ from $x = 0$ to $x = 1$.



$$\int_0^1 \pi (\sqrt{x})^2 \, dx$$

$$= \int_0^1 \pi x \, dx = \pi \cdot \frac{1}{2} x^2 \Big|_0^1 = \frac{\pi}{2}$$

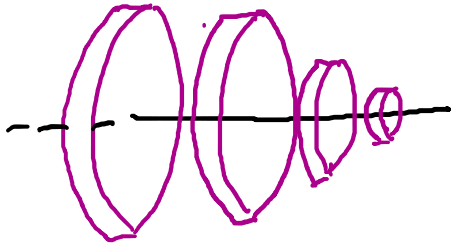
A: $\pi - 1$
B: $\pi/2$
C: $\pi/3$
D: π
E: None

Other Volume Integrals

- Integrating disc volumes along an axis

$$\lim_{n\to\infty} \sum_1^n \pi\big(f(x_i)\big)^2 \Delta x$$

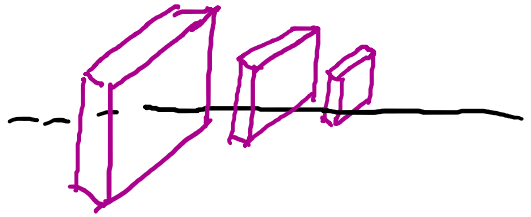
$$= \int \pi \, (f(x))^2 \, dx$$



- What about other shapes?

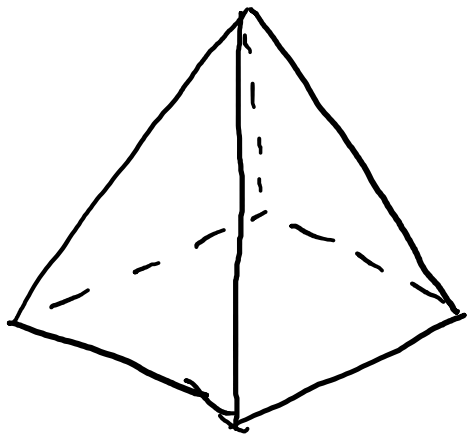
$\lim_{n\to\infty} \sum_1^n A(x)\Delta x$, where $A(x)$ is the area of each slice to be multiplied by Δx.

$$= \int A(x)\, dx$$



Example - Pyramid

$$\int_a^b f(x)\,dx = F(b) - F(a) = -\left(F(a) - F(b)\right) = -\int_b^a f(x)\,dx$$



- Suppose the vertical cross section of a pyramid 100 meters tall is always a square, and suppose the side-length of the square is $100 - x$ meters, where x is the height above ground in meters.
- What is the volume of the pyramid?

$$A(x) = (100 - x)^2$$

$$Vol = \int_{x=0}^{x=100} (100 - x)^2\,dx$$

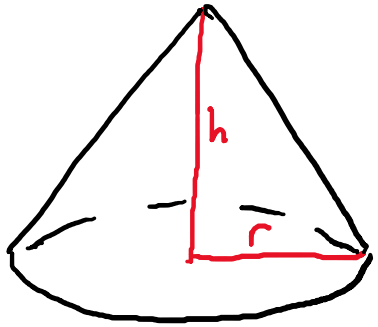
Let $u = 100 - x$

$du = -dx$

$$= \int_{u=100}^{u=0} -u^2\,du = \int_{u=0}^{u=100} u^2\,du$$

$$= \frac{1}{3}u^3\Big|_{u=0}^{u=100} = \frac{1}{3}\cdot 100^3 \approx 333333\ m^3$$

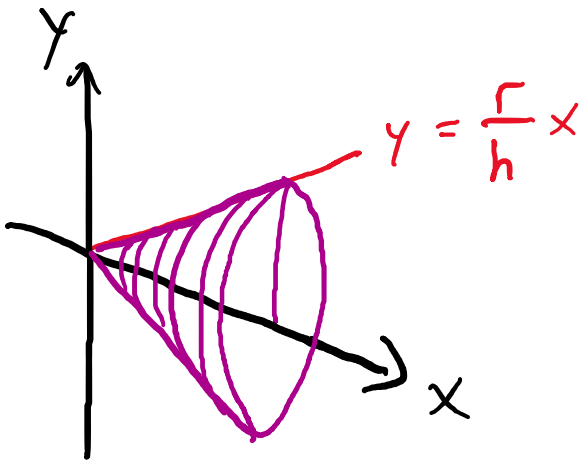
Surface Areas?



$$Vol = \int_0^h \pi (f(x))^2 \, dx$$

Surface area (not incl base) $\stackrel{?}{=} \int_0^h 2\pi (f(x)) \, dx$

A: True
B: False
C: ???
D: !!!
E: None



FALSE

Correct Surface Area

$$= \int_0^h 2\pi (f(x)) \sqrt{1 + \left(\frac{d}{dx} f(x)\right)^2} \, dx$$

- Using discs does NOT work for surface areas because you are incorrectly approximating the paths.