Improper integrals
Lecture 2c – 2021-05-19

MAT A35 – Summer 2021 – UTSC

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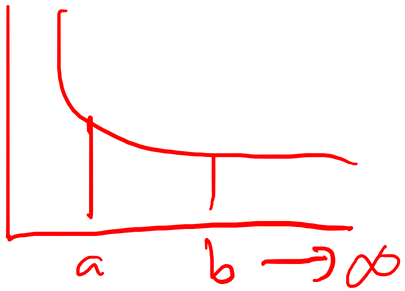
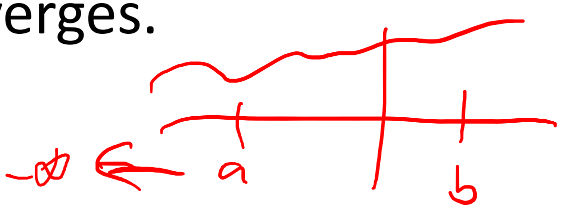
Integrating to infinity



- $\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$
  - If the limit exists, then the improper integral converges.
  - If the limit does not exist, then the integral diverges.

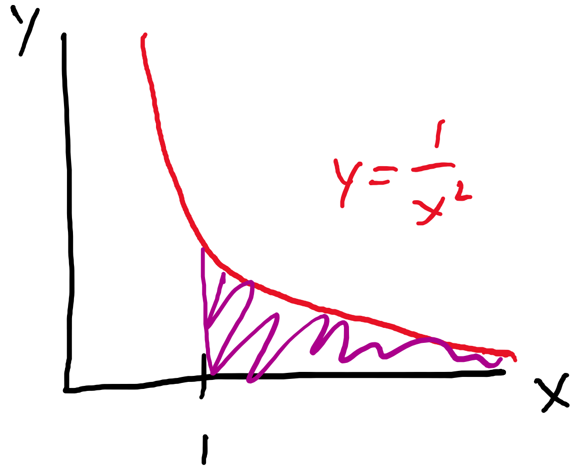


- $\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$
  - If the limit exists, then the improper integral converges.
  - If the limit does not exist, then the integral diverges.
- $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx,$
  where c is any real number.
  - If both integrals on the right converge, then so does the integral on the left.

Convergent Example

$$\int \frac{1}{x^2}\,dx = -\frac{1}{x} + C$$

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx$$



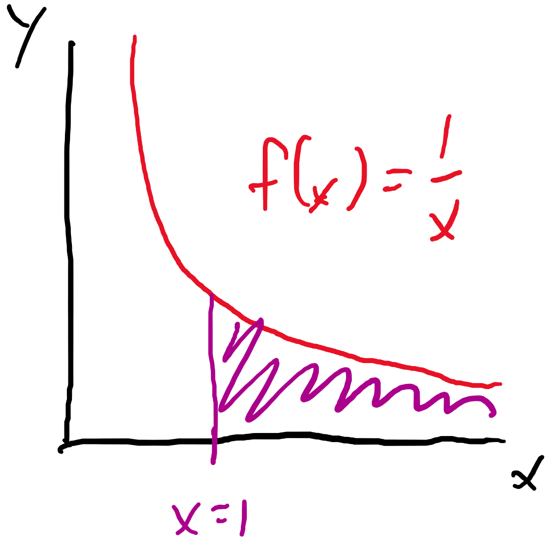
$$= \lim_{b\to\infty} \left[-\frac{1}{x}\right]\Bigg|_{x=1}^{x=b}$$

$$= \lim_{b\to\infty} \left[\cancel{-\frac{1}{b}} + 1\right]$$

$$\left[-\frac{1}{b}\right] - \left[-\frac{1}{1}\right]$$

$$= 1$$

Divergent Example



$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx$$

$$= \lim_{b\to\infty} \left[\ln|x|\right]\Big|_{x=1}^{x=b}$$

$$= \lim_{b\to\infty} \left[\ln b - \underbrace{\ln 1}_{=0}\right]$$

$$= \lim_{b\to\infty} \ln b = \infty$$

Example

$$\int_{-\infty}^{\infty} \frac{x}{(x^2+1)^2}\,dx = \underline{\int_{-\infty}^{0}\frac{x}{(x^2+1)^2}\,dx} + \overbrace{\int_{0}^{\infty}\frac{x}{(x^2+1)^2}\,dx}$$

$$\int_{-\infty}^{0}\frac{x}{(x^2+1)^2}\,dx = \lim_{a\to-\infty}\int_{x=a}^{x=0}\frac{x}{(x^2+1)^2}\,dx = \lim_{a\to-\infty}\int_{u=a^2+1}^{u=1}\frac{\frac{1}{2}}{u^2}\,du$$

$u = x^2+1$

$du = 2x\,dx$

$$= \lim_{a\to-\infty}\frac{1}{2}\left[-\frac{1}{u}\right]\Bigg|_{u=a^2+1}^{u=1} = \lim_{a\to-\infty}\frac{1}{2}\left[-1+\cancel{\frac{1}{a^2+1}}\right] = -\frac{1}{2}$$

$\underline{\text{Alternately}}$ $$\int\frac{x}{(x^2+1)^2}\,dx = \int\frac{\frac{1}{2}}{u^2}\,du = \frac{1}{2}\left[-\frac{1}{u}\right]+C = -\frac{1}{2(x^2+1)}+C$$

$$\Rightarrow \lim_{a\to-\infty}\int_{a}^{0}\frac{x}{(x^2+1)^2}\,dx = \lim_{a\to-\infty}\left[-\frac{1}{2(x^2+1)}\right]\Bigg|_{x=a}^{x=0} = \lim_{a\to-\infty}\left[-\frac{1}{2}+\frac{1}{2(a^2+1)}\right] = -\frac{1}{2}$$

Example (continued)

$$\int_0^\infty \frac{x}{(x^2+1)^2}\,dx = \lim_{b\to\infty}\int_0^b \frac{x}{(x^2+1)^2}\,dx = \lim_{b\to\infty}\int_1^{b^2+1} \frac{\frac{1}{2}}{u^2}\,du$$

$u = x^2+1$

$du = 2x\,dx$

$$= \lim_{b\to\infty}\left[\frac{1}{2}\cdot -\frac{1}{u}\right]\Bigg|_{u=1}^{u=b^2+1}$$

$$= \lim_{b\to\infty}\left[-\frac{1}{2}\cdot \cancelto{0}{\frac{1}{(b^2+1)}} + \frac{1}{2}\right] = \frac{1}{2}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x}{(x^2+1)^2}\,dx = -\frac{1}{2} + \frac{1}{2} = 0$$

Try it out

- $\int_0^\infty e^{-x}\,dx$

$$= \lim_{b\to\infty} \int_0^b e^{-x}\,dx = \lim_{b\to\infty} \left[-e^{-x}\right]\Big|_0^b$$

$$= \lim_{b\to\infty} \left[-e^{-b}+1\right] = 1$$

$$\lim_{b\to\infty} e^{-b} = 0$$

A: Convergent
B: Divergent
E: Neither

- $\int_{-\infty}^\infty e^{x}\,dx$

$$= \int_{-\infty}^0 e^x\,dx + \underbrace{\int_0^\infty e^x\,dx}$$

$$\left[e^x\right]\Big|_0^\infty = e^\infty - e^0 = \infty - 1 = \infty$$

A: Convergent
B: Divergent
E: Neither

Application to Radioactive Decay

- Plutonium-238 has a half-life of 87.7 years.
- You have a sample of plutonium that releases 100 rem/yr.
- How much total rems of radioactivity is emitted by the sample over eternity?

Radioactive decay: $P(t) = P(0)\,e^{-kt}$ $\Rightarrow$ $-87.7k = \ln\frac{1}{2}$

$$\frac{1}{2}P(0) = P(0)\,e^{-87.7k}$$

$$k = 0.0079$$

REMs released: $\int_0^\infty p(t)\cdot r\,dt$, where r is a rate of REMs released per unit of plutonium. $P(0)\cdot r = 100$

$$= \int_0^\infty P(0)\cdot r\cdot e^{-kt}\,dt = \int_0^\infty 100e^{-kt}\,dt = \left[-\frac{1}{k}\cdot 100e^{-kt}\right]_0^\infty$$

$$= \lim_{b\to\infty}\left[-\frac{100}{k}\left(-1+e^{-bk}\right)\right] = \frac{100}{k} = 12658 \text{ rems}$$

Reminder: L'Hopital's Rule

- If $f(x)$ and $g(x)$ are differentiable on an open interval I except possibly at a point $c \in I$ (c is allowed to be $\pm\infty$), and if $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0 \text{ or } \pm\infty$, and $g'(x) \neq 0$ (except possibly at $g'(c)$) then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

- Lemma: If $r > 0$ and n is any number, then

$$\lim_{x \to \infty} x^n e^{-rx} = \lim_{x \to \infty} \frac{x^n}{e^{rx}} = 0$$

Example: $\int_0^\infty xe^{-x}\,dx$

DETAIL

$\boxed{\int u\,dv = uv - \int v\,du}$

Solve antiderivative first

$\int xe^{-x}\,dx$

$u = x \qquad v = -e^{-x}$

$du = dx \qquad dv = e^{-x}\,dx$

$= -xe^{-x} + \int (+e^{-x})\,dx$

$= -xe^{-x} - e^{-x} + C$

$\Rightarrow \int_0^\infty xe^{-x}\,dx = \lim_{b\to\infty} \left[-xe^{-x} - e^{-x}\right]\Big|_0^b$

$= \lim_{b\to\infty} \left[\left(\cancel{-be^{-b}} - \cancel{e^{-b}}\right) - (0 - 1)\right]$

$= 1$

Integration Review

- $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$, where $\Delta x = \frac{1}{b-a}$ and $x_i = a + i\Delta x$.
- $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)dx$
- $\int_a^b f(x)\,dx + \int_a^b g(x)\,dx = \int_a^b [f(x) + g(x)]dx$
- If $f(x) > 0$ for all $x \in [a, b]$, then $\int_a^b f(x)\,dx$ is the area under the curve $y = f(x)$ in the interval $[a, b]$
- FTC: If f is continuous and $\int f(x)dx = F(x)$, then $\int_a^b f(x)dx = F(b) - F(a)$. Also, $\frac{d}{dx}\int_a^x f(t)dt = f(x)$.
- U-substitution: $\int_{x=a}^{x=b} f'(u) \cdot u'dx = f(u)\Big|_{u=u(a)}^{u=u(b)}$
- Integration by parts (DETAIL): $\int udv = uv - \int vdu$.
- Volume = $\int_a^b A(x)\,dx$, and $A(x) = \pi\big(f(x)\big)^2$ for a solid of revolution