Improper integrals
Lecture 2c – 2021-05-19

MAT A35 – Summer 2021 – UTSC

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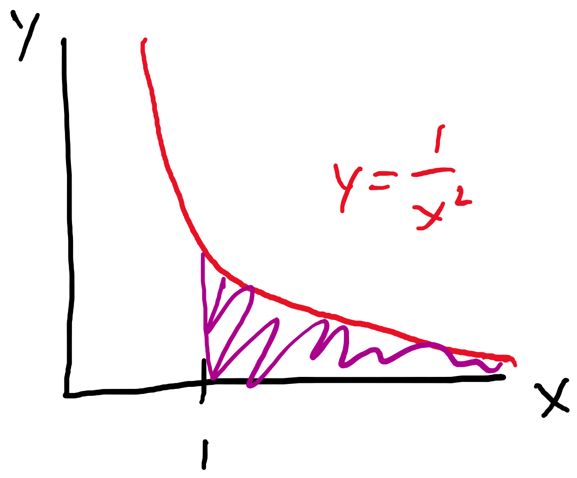
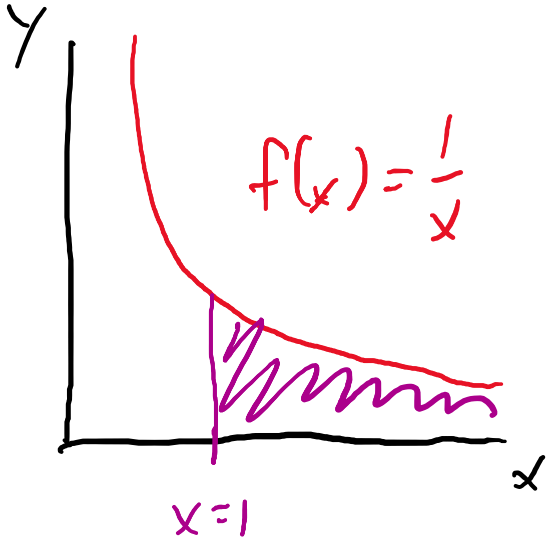
Integrating to infinity

- $\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$
  - If the limit exists, then the improper integral converges.
  - If the limit does not exist, then the integral diverges.
- $\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$
  - If the limit exists, then the improper integral converges.
  - If the limit does not exist, then the integral diverges.
- $\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx,$
  where c is any real number.
  - If both integrals on the right converge, then so does the integral on the left.

Convergent Example



Divergent Example



Example

Example (continued)

Try it out

- $\int_0^\infty e^{-x}\,dx$

A: Convergent
B: Divergent
E: Neither

- $\int_{-\infty}^\infty e^{x}\,dx$

A: Convergent
B: Divergent
E: Neither

Application to Radioactive Decay

- Plutonium-238 has a half-life of 87.7 years.
- You have a sample of plutonium that releases 100 rem/yr.
- How much total rems of radioactivity is emitted by the sample over eternity?

Reminder: L'Hopital's Rule

- If $f(x)$ and $g(x)$ are differentiable on an open interval I except possibly at a point $c \in I$ (c is allowed to be $\pm\infty$), and if $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0 \text{ or } \pm\infty$, and $g'(x) \neq 0$ (except possibly at $g'(c)$) then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

- Lemma: If $r > 0$ and n is any number, then

$$\lim_{x\to\infty} x^n e^{-rx} = \lim_{x\to\infty} \frac{x^n}{e^{rx}} = 0$$

Example: $\int_0^\infty xe^{-x}\,dx$

Integration Review

- $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^n f(x_i)\Delta x$, where $\Delta x = \frac{1}{b-a}$ and $x_i = a + i\Delta x$.
- $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)dx$
- $\int_a^b f(x)\,dx + \int_a^b g(x)\,dx = \int_a^b [f(x) + g(x)]dx$
- If $f(x) > 0$ for all $x \in [a,b]$, then $\int_a^b f(x)\,dx$ is the area under the curve $y = f(x)$ in the interval $[a,b]$
- FTC: If f is continuous and $\int f(x)dx = F(x)$, then $\int_a^b f(x) = F(b) - F(a)$. Also, $\frac{d}{dx}\int_a^x f(t)dt = f(x)$.
- U-substitution: $\int_{x=a}^{x=b} f'(u)\cdot u'dx = f(u)\big|_{u=u(a)}^{u=u(b)}$
- Integration by parts (DETAIL): $\int udv = uv - \int vdu$.
- Volume = $\int_a^b A(x)\,dx$, and $A(x) = \pi\big(f(x)\big)^2$ for a solid of revolution