

Matrix Operations

Lecture 3a – 2021-05-26

MAT A35 – Summer 2021 – UTSC

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Scalars to Vectors

- A scalar is a single number $a \in \mathbb{R}$.

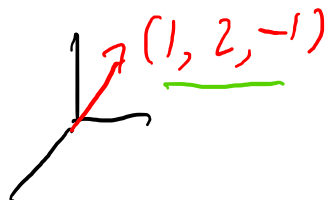
Ex. UTSC is 57 years old

↙ size of vector

- A vector is a collection of n numbers $v = \{v_1, v_2, \dots, v_n\} \in \mathbb{R}^n$.

Ex. UTSC's GPS coordinates are $(\underbrace{43.7830}_{N/S}, \underbrace{-77.1874}_{E/W})$

Ex. A bird population has 100 hatchlings & 200 adults.
Can write as a population vector $(100, 200)$

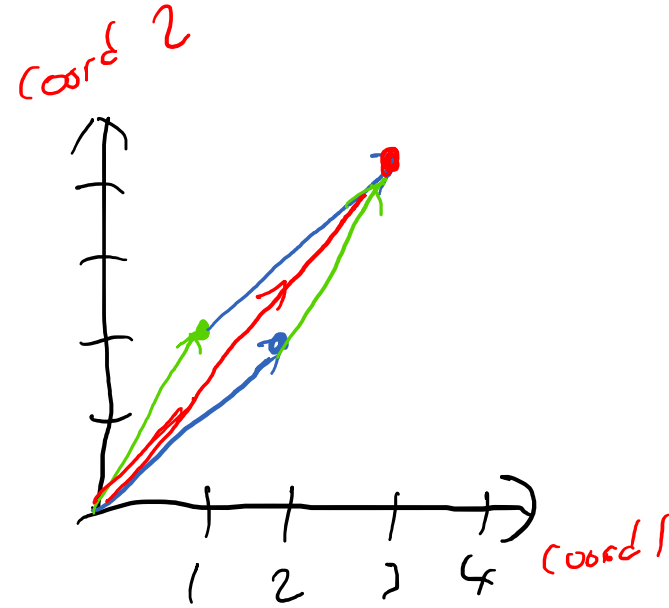
Ex.  A point or direction & magnitude in 3D.

Operations on Vectors

- Addition: only works on same size vectors.

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix}$$

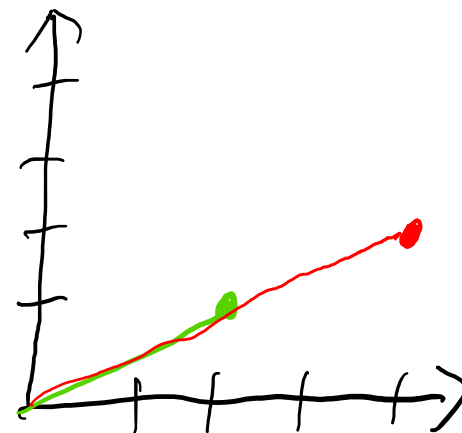
Ex. $\begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$



- Multiplication by Scalar

$$a \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a b_1 \\ a b_2 \end{bmatrix}$$

Ex. $2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$



Dot Product on Vectors

- Given two vectors of the same size $a, b \in \mathbb{R}^n$, where $a = \{a_1, \dots, a_n\}$ and $b = \{b_1, \dots, b_n\}$, the dot product $a \cdot b = a_1b_1 + a_2b_2 + \dots + a_nb_n$, a scalar.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix} = -1 + 8 + 6 = 13$$

- The dot product intuitively tells you how similar two vectors are.

Try it out

$$\bullet \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \end{pmatrix} = -2 + 2 = 0$$

A: $\begin{pmatrix} -2 \\ 2 \end{pmatrix}$, B: 0, C: -3, D: $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$, E: None

$$\bullet \begin{pmatrix} -2 \\ 5 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

A: $\begin{pmatrix} -1 \\ 7 \end{pmatrix}$, B: $\begin{pmatrix} -1 \\ 7 \\ 3 \end{pmatrix}$, C: 8, D: 0, E: None

$$\bullet \underbrace{0}_{\text{row}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - 2 \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

A: $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, B: $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, C: $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, D: 0, E: None

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \cdot 0 \\ 2 \cdot 0 \\ 2 \cdot 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Vectors to Matrices

- A matrix is a rectangular grid of scalars, with several important properties.
 - A matrix $A = [a_{ij}]$ with size $m \times n$ has m rows and n columns, and a_{ij} represents the entry in the i th row and j th column.
 - Like vectors, can add matrices, and multiply by a scalar.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \text{ is } 3 \times 2 \text{ matrix, and } a_{21} = 3$$

$$2A = \begin{bmatrix} 2 & 4 \\ 6 & 8 \\ 10 & 12 \end{bmatrix} \quad A + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 3 & 5 \\ 6 & 6 \end{bmatrix}$$

Matrix transposition

- Let a matrix $A = [a_{ij}]$ with size $m \times n$ has m rows and n columns, where a_{ij} represents the entry in the i th row and j th column.
- Then the transpose $B = [b_{ij}] = A^T$ has size $n \times m$, which means it has n rows and m columns, and $b_{ij} = a_{ji}$.
- Transposition flips all terms across the diagonal line from the top-left to the bottom-right.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \end{bmatrix}^T = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

3×2 2×3

Row and column matrices = vectors

- A matrix with just 1 row is a row vector.

$[1 \ 2 \ 3]$ is a row vector

- A matrix with just 1 column is a column vector. Normally, when we say vector, we will refer to a column vector.

$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is a column vector.

Matrix Products

- Let A be a $m \times n$ matrix and let B be a $n \times p$ matrix. Then the product $C = AB$ is a $m \times p$ matrix such that

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}$$

- Alternately, can think of A as a collection of m stacked row vectors, and B as a collection of p column vectors. Then c_{ij} is the dot product of the i th row and the j th column, as vectors.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} -3 & -2 \\ -1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} (1)(-3) + (2)(-1) + (3)(1) & (1)(-2) + (2)(0) + (3)(2) \\ (4)(-3) + (5)(-1) + (6)(2) & (4)(-2) + (5)(0) + (6)(2) \end{bmatrix}$$

2×3 3×2
—————
 2×2

$$= \begin{bmatrix} -2 & 4 \\ -11 & 7 \end{bmatrix}$$

Try it out

$$\bullet \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} [4 \quad 5 \quad 6]$$

A: 32

B: $\begin{bmatrix} 4 \\ 10 \\ 18 \end{bmatrix}$

C: [4 10 18]

D: $\begin{bmatrix} 4 & 5 & 6 \\ 8 & 10 & 12 \\ 12 & 15 & 18 \end{bmatrix}$

E: None

$$\bullet \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

A: $\begin{bmatrix} 17 \\ 39 \end{bmatrix}$

B: $\begin{bmatrix} 5 & 10 \\ 18 & 24 \end{bmatrix}$

C: [23 34]

D: 56

E: None

Matrices are transformations of vectors

- Scaling operators: $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$

- Stretching/squashing: $\begin{bmatrix} 0.5 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.5x \\ 2y \end{bmatrix}$

- Rotations: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$

- Etc...

Application: Leslie Matrices



- Consider a rabbit population that can be divided into two age classes: young and adult.
 - Young rabbits can reach sexual maturity within 6-7 months.
 - Adult rabbits live on average for 9 years.
- Let's consider a *state vector* of the rabbit population:
 - $\begin{bmatrix} \text{number of young rabbits} \\ \text{number of adult rabbits} \end{bmatrix}$
- Each year, both the young and adult rabbits have a chance of surviving (survivability), and a number of offspring (fecundity), which we can encode in a *Leslie matrix*.
 - $\begin{bmatrix} \text{average fecundity of young} & \text{average fecundity of adults} \\ \text{survivability of young} & \text{survivability of adults} \end{bmatrix}$

Rabbit population - continued



$$\begin{bmatrix} \text{average fecundity of young} & \text{average fecundity of adults} \\ \text{survivability of young} & \text{survivability of adults} \end{bmatrix} \begin{bmatrix} \text{number of young rabbits} \\ \text{number of adult rabbits} \end{bmatrix}$$

$$= \begin{bmatrix} \text{number of offspring of young} + \text{number of offspring of adults} \\ \text{number of surviving young} + \text{number of surviving adults} \end{bmatrix}$$

$$= \begin{bmatrix} \text{number of young rabbits the following year} \\ \text{number of adult rabbits the following year} \end{bmatrix}$$