Matrix Operations
Lecture 3a – 2021-05-26

MAT A35 – Summer 2021 – UTSC

Prof. Yun William Yu

Scalars to Vectors

- A scalar is a single number $a \in \mathbb{R}$.

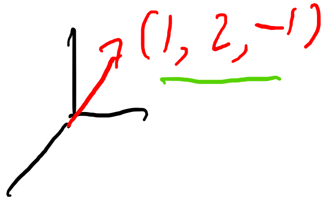
Ex. UTSC is 57 years old

size of vector

- A vector is a collection of n numbers $v = \{v_1, v_2, \ldots, v_n\} \in \mathbb{R}^n$.

Ex. UTSC's GPS coordinates are $(\underbrace{43.7830}_{N/S}, \underbrace{-79.1874}_{E/W})$

Ex. A bird population has 100 hatchlings & 200 adults.
Can write as a population vector $(100, 200)$

Ex. 

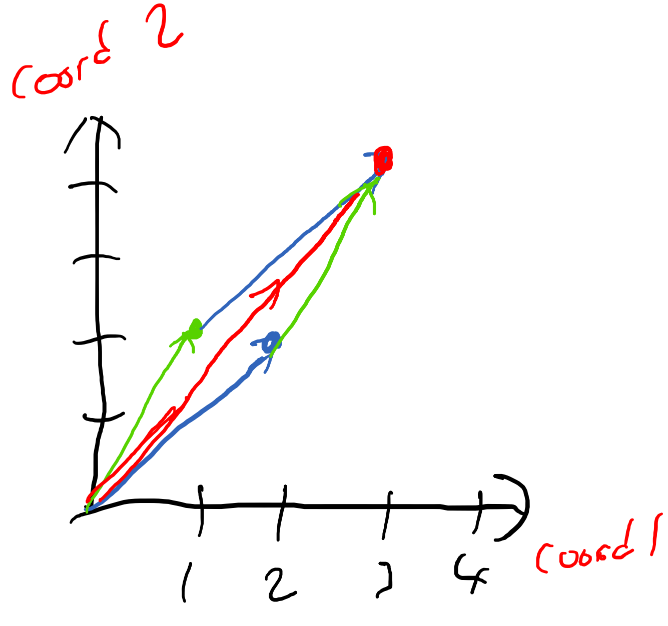
A point or direction & magnitude in 3D.

Operations on Vectors

- Addition: only works on same size vectors.

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix}$$

Ex. $\begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$



- Multiplication by Scalar

$$a \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a\,b_1 \\ a\,b_2 \end{bmatrix}$$

Ex. $2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

Dot Product on Vectors

- Given two vectors of the same size $a, b \in \mathbb{R}^n$, where $a = \{a_1, \ldots, a_n\}$ and $b = \{b_1, \ldots, b_n\}$, the dot product $a \cdot b = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$, a scalar.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix} = -1 + 8 + 6 = 13$$

- The dot product intuitively tells you how similar two vectors are.

Try it out

- $\begin{pmatrix}1\\2\end{pmatrix}\cdot\begin{pmatrix}-2\\1\end{pmatrix} = -2+2=0$

A: $\begin{pmatrix}-2\\2\end{pmatrix}$, B: 0, C: -3, D: $\begin{pmatrix}-1\\3\end{pmatrix}$, E: None

- $\begin{pmatrix}-2\\5\end{pmatrix}+\begin{pmatrix}1\\2\\3\end{pmatrix}$

A: $\begin{pmatrix}-1\\7\end{pmatrix}$, B: $\begin{pmatrix}-1\\7\\3\end{pmatrix}$, C: 8, D: 0, E: None

- $\underbrace{0\begin{pmatrix}1\\2\\3\end{pmatrix}}_{} - 2\begin{pmatrix}0\\0\\0\end{pmatrix}$

A: $\begin{pmatrix}0\\0\\0\end{pmatrix}$, B: $\begin{pmatrix}1\\2\\3\end{pmatrix}$, C: $\begin{pmatrix}-1\\0\\1\end{pmatrix}$, D: 0, E: None

$$\begin{pmatrix}0\\0\\0\end{pmatrix} - \begin{pmatrix}2\cdot 0\\2\cdot 0\\2\cdot 0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} - \begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Vectors to Matrices

- A matrix is a rectangular grid of scalars, with several important properties.
  - A matrix $A = [a_{ij}]$ with size $m \times n$ has m rows and n columns, and a_{ij} represents the entry in the ith row and jth column.
  - Like vectors, can add matrices, and multiply by a scalar.

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ is 3×2 matrix, and $a_{21} = 3$

$2A = \begin{bmatrix} 2 & 4 \\ 6 & 8 \\ 10 & 12 \end{bmatrix}$ $\quad A + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 3 & 5 \\ 6 & 6 \end{bmatrix}$

Matrix transposition

- Let a matrix $A = [a_{ij}]$ with size $m \times n$ has m rows and n columns, where a_{ij} represents the entry in the ith row and jth column.
- Then the transpose $B = [b_{ij}] = A^T$ has size $n \times m$, which means it has n rows and m columns, and $b_{ij} = a_{ji}$.
- Transposition flips all terms across the diagonal line from the top-left to the bottom-right.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^T}_{3\times 2} = \underbrace{\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}}_{2\times 3}.$$

$$\begin{bmatrix} 1 & 2 \end{bmatrix}^T = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Row and column matrices = vectors

- A matrix with just 1 row is a row vector.

$[1 \quad 2 \quad 3]$ is a row vector

- A matrix with just 1 column is a column vector. Normally, when we say vector, we will refer to a column vector.

$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is a column vector.

Matrix Products

- Let A be a $m \times n$ matrix and let B be a $n \times p$ matrix. Then the product $C = AB$ is a $m \times p$ matrix such that
$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots a_{ik}b_{kj}$$
- Alternately, can think of A as a collection of m stacked row vectors, and B as a collection of p column vectors. Then c_{ij} is the dot product of the ith row and the jth column, as vectors.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} -3 & -2 \\ -1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} (1)(-3)+(2)(-1)+(3)(1) & (1)(-2)+(2)(0)+(3)(2) \\ (4)(-3)+(5)(-1)+(6)(2) & (4)(-2)+(5)(0)+(6)(2) \end{bmatrix}$$

2×3 $\quad$ 3×2

2×2

$$= \begin{bmatrix} -2 & 4 \\ -11 & 4 \end{bmatrix}$$

Try it out

- $\begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}4 & 5 & 6\end{bmatrix}$

$\underbrace{3\times 1 \quad 1\times 3}$

$= 3\times 3$

A: 32

B: $\begin{bmatrix}4\\10\\18\end{bmatrix}$

C: $\begin{bmatrix}4 & 10 & 18\end{bmatrix}$

D: $\begin{bmatrix}4 & 5 & 6\\8 & 10 & 12\\12 & 15 & 18\end{bmatrix}$

E: None

$\begin{bmatrix}1 & 2 & 3\end{bmatrix}\begin{bmatrix}4\\5\\6\end{bmatrix} = 1\cdot 4 + 2\cdot 5 \cdot 3\cdot 6$

$= 32$

- $\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}5\\6\end{bmatrix}$

$2\times 2 \quad 2\times 1$

2×1

A: $\begin{bmatrix}17\\39\end{bmatrix}$

B: $\begin{bmatrix}5 & 10\\18 & 24\end{bmatrix}$

C: $\begin{bmatrix}23 & 34\end{bmatrix}$

D: 56

E: None

$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}5\\6\end{bmatrix}$

$= \begin{bmatrix}1\cdot 5 + 2\cdot 6\\3\cdot 5 + 4\cdot 6\end{bmatrix} = \begin{bmatrix}17\\39\end{bmatrix}$

Outer products

Inner product

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$\Longleftrightarrow \underset{1\times 3}{\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}} = \begin{bmatrix} a_1 b_1 + a_2 b_2 + a_3 b_3 \end{bmatrix}$$

Outer product

$$\underset{3\times 1}{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}} \underset{1\times 3}{\begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}} = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix}$$

Matrix multiplication using outer products

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} -3 & -2 \\ -1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} (1)(-3)+(2)(-1)+(3)(1) & (1)(-2)+(2)(0)+(3)(2) \\ (4)(-3)+(5)(-1)+(6)(1) & (4)(-2)+(5)(0)+(6)(2) \end{bmatrix}$$

2×3 $\quad 3 \times 2$

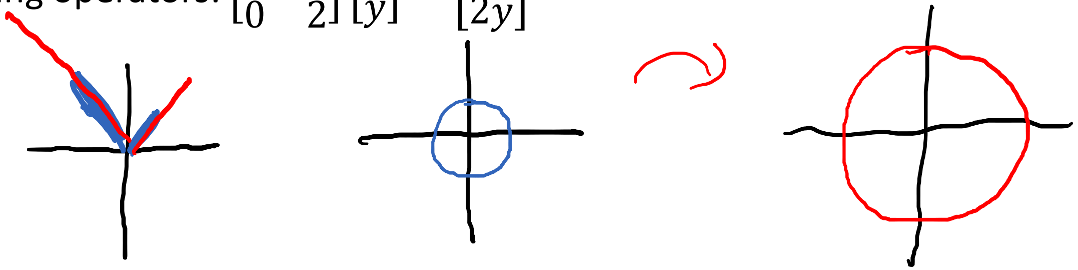
$$= \begin{bmatrix} -2 & 4 \\ -11 & 4 \end{bmatrix} \quad 2 \times 2$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} -3 & -2 \\ -1 & 0 \\ 1 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 4 \end{bmatrix} \begin{bmatrix} -3 & -2 \end{bmatrix}}_{2\times 1 \quad 1\times 2} + \underline{\begin{bmatrix} 2 \\ 5 \end{bmatrix} \begin{bmatrix} -1 & 0 \end{bmatrix}} + \begin{bmatrix} 3 \\ 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix}$$

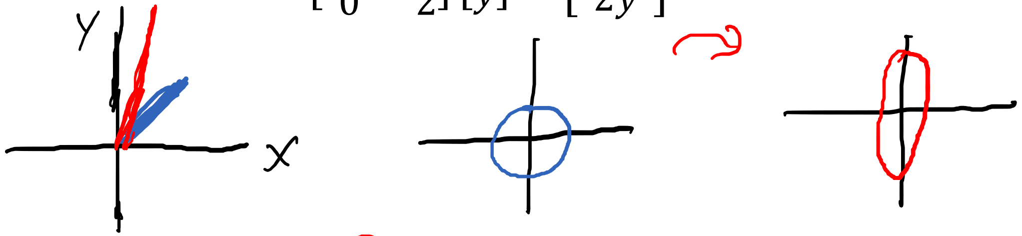
$$= \begin{bmatrix} -3 & -2 \\ -12 & -8 \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ -5 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -11 & 4 \end{bmatrix}$$

Matrices are transformations of vectors

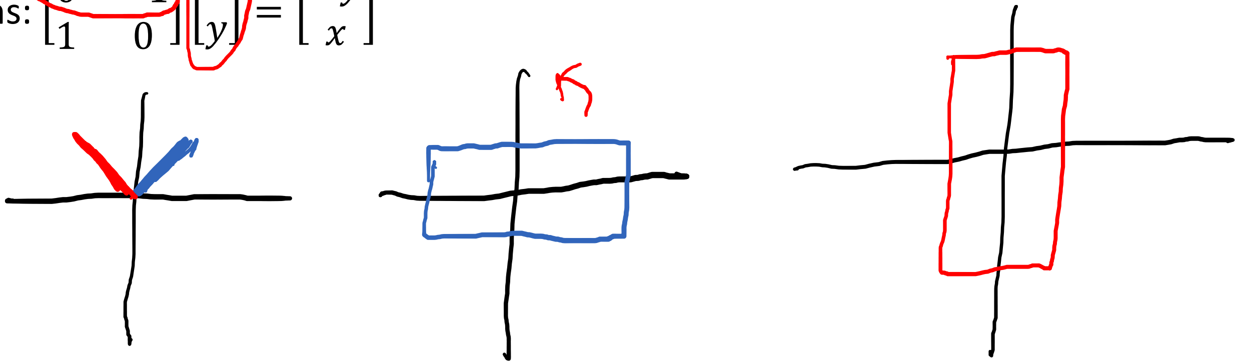
- Scaling operators: $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$



- Stretching/squashing: $\begin{bmatrix} 0.5 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.5x \\ 2y \end{bmatrix}$



- Rotations: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$



- Etc...

Application: Leslie Matrices

- Consider a rabbit population that can be divided into two age classes: young and adult.
  - Young rabbits can reach sexual maturity within 6-7 months.
  - Adult rabbits live on average for 9 years.
- Let's consider a *state vector* of the rabbit population:
  - $\begin{bmatrix} \text{number of young rabbits} \\ \text{number of adult rabbits} \end{bmatrix}$
- Each year, both the young and adult rabbits have a chance of surviving (survivability), and a number of offspring (fecundity), which we can encode in a *Leslie matrix*.
  - $\begin{bmatrix} \text{average fecundity of young} & \text{average fecundity of adults} \\ \text{survivability of young} & \text{survivability of adults} \end{bmatrix}$

Rabbit population - continued

$$\begin{bmatrix} \text{average fecundity of young} & \text{average fecundity of adults} \\ \text{survivability of young} & \text{survivability of adults} \end{bmatrix} \begin{bmatrix} \text{number of young rabbits} \\ \text{number of adult rabbits} \end{bmatrix}$$

$$= \begin{bmatrix} \text{number of offspring of young} + \text{number of offspring of adults} \\ \text{number of surviving young} + \text{number of surviving adults} \end{bmatrix}$$

$$= \begin{bmatrix} \text{number of young rabbits the following year} \\ \text{number of adult rabbits the foollowing year} \end{bmatrix}$$

Ex

$$L = \begin{bmatrix} 2 & 3 \\ 0.5 & 0.9 \end{bmatrix}$$

Year 1: $\begin{array}{l} \text{young:} \\ \text{adults:} \end{array} \overbrace{\begin{bmatrix} 100 \\ 10 \end{bmatrix}}^{p}$

Year 2: $Lp = \begin{bmatrix} 2 & 3 \\ 0.5 & 0.9 \end{bmatrix} \begin{bmatrix} 100 \\ 10 \end{bmatrix} = \begin{bmatrix} 230 \\ 59 \end{bmatrix}$

Year 3: $\overbrace{\begin{bmatrix} 2 & 3 \\ 0.5 & 0.9 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0.5 & 0.9 \end{bmatrix} \begin{bmatrix} 100 \\ 10 \end{bmatrix}}^{L^2p} = \overbrace{\begin{bmatrix} 2 & 3 \\ 0.5 & 0.9 \end{bmatrix}}^{L} \overbrace{\begin{bmatrix} 230 \\ 59 \end{bmatrix}}^{Lp} = \overbrace{\begin{bmatrix} 637 \\ 168.1 \end{bmatrix}}^{L^2p}$

Leslie Diagrams

- We can encode a Leslie matrix as a graph

Row 1, col 1

row 1 col 2

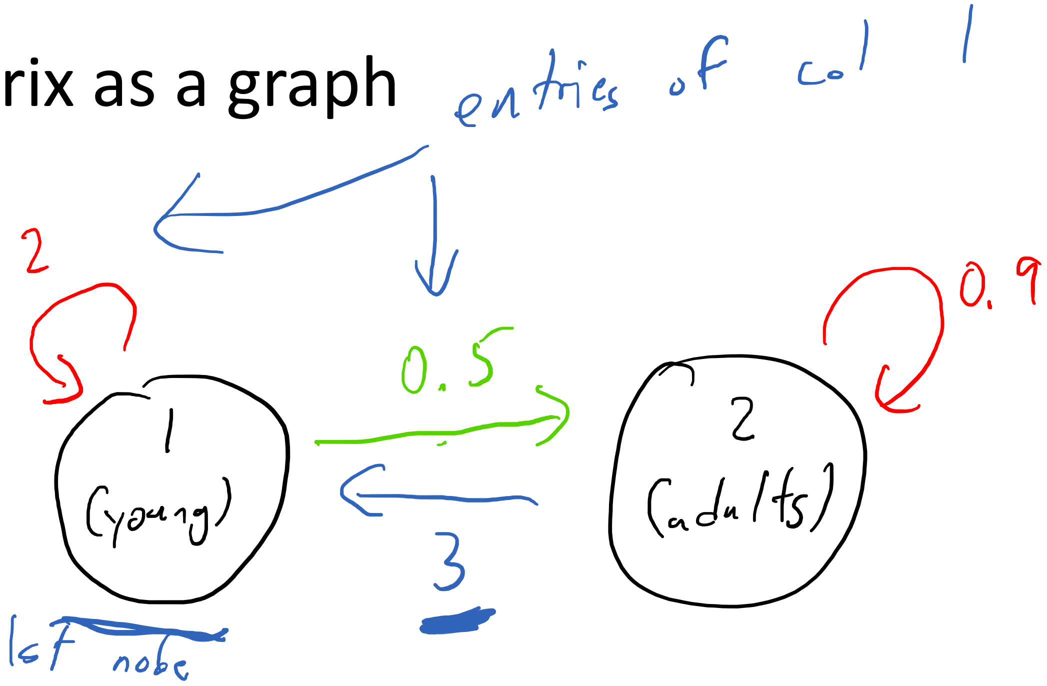
2 → 1

$$L = \begin{bmatrix} 2 & 3 \\ 0.5 & 0.9 \end{bmatrix}$$

row 2, col 1

1 → 2

1st col

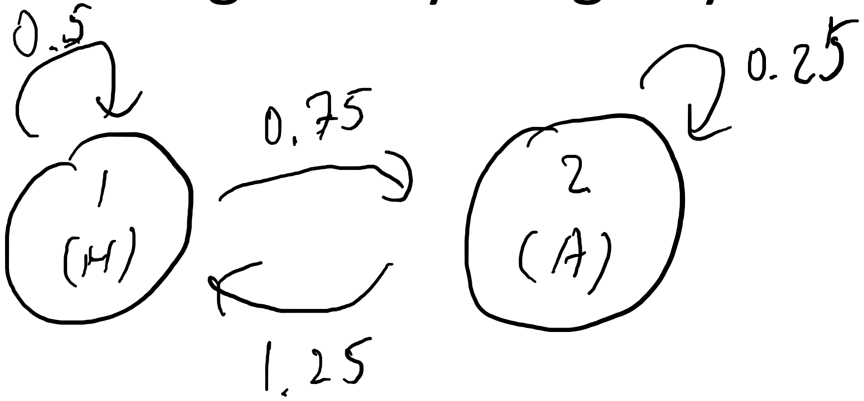


Number of young next year = 2 · [young] + 3 · [adults]

Number of adults next year = 0.5 [young] + 0.9 [adults]

Try it out

- A population of birds has the following Leslie diagram with 100 hatchlings (H) and 40 adults (A) in year 1. Estimate the number of hatchlings and young in year 2.



$$LP = \begin{bmatrix} 50+50 \\ 75+10 \end{bmatrix} = \begin{bmatrix} 100 \\ 85 \end{bmatrix}$$

$$L = \begin{bmatrix} 0.5 & 1.25 \\ 0.75 & 0.25 \end{bmatrix} \qquad P = \begin{bmatrix} 100 \\ 40 \end{bmatrix}$$

A: 100 hatchlings, 40 adults
B: 100 hatchlings, 85 adults
C: 200 hatchlings, 40 adults
D: 200 hatchlings, 85 adults
E: None

Identity matrix

- Identity matrix: a special $n \times n$ matrix $I_n = I$ such that $AI = A$ for any $m \times n$ matrix and $IA = A$ for any $n \times m$ matrix.
  - $I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$ has 1's along the diagonal and 0's elsewhere.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

Matrix algebra

$$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}17\\39\end{bmatrix} = \begin{bmatrix}17\\-39\end{bmatrix}$$

- $A(BC) = (AB)C$

$$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}5\\6\end{bmatrix}$$

$$\begin{bmatrix}1 & 2\\-3 & -4\end{bmatrix}\begin{bmatrix}5\\6\end{bmatrix} = \begin{bmatrix}17\\-39\end{bmatrix}$$

- $A(B + C) = AB + AC$

$$\begin{bmatrix}3\\2\end{bmatrix} + \begin{bmatrix}7\\4\end{bmatrix} = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\left(\begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}3\\4\end{bmatrix}\right) = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}4\\6\end{bmatrix} = \begin{bmatrix}10\\6\end{bmatrix}$$

- $(B + C)A = BA + CA$

- $AB \neq BA$ (in general)

$$\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix} = \begin{bmatrix}2 & 0\\0 & 0\end{bmatrix} \neq \begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix} = \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$$

Try it out

- $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}\right)$

A: $\begin{bmatrix} 3 & 5 \\ 5 & 9 \end{bmatrix}$
B: $\begin{bmatrix} 2 & 5 \\ 5 & 8 \end{bmatrix}$
C: $\begin{bmatrix} 5 & 11 \\ 11 & 25 \end{bmatrix}$
D: $\begin{bmatrix} 6 & 11 \\ 11 & 26 \end{bmatrix}$
E: None

- $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\left(\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}}\right)$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

A: $\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$
B: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
C: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
D: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
E: None