Matrix inverses and determinants
Lecture 3c – 2021-05-28

MAT A35 – Summer 2021 – UTSC

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“Dividing” by a matrix

Addition :
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$$

Subtraction :
$$\begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Multiplication :
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 2+12 & 4+16 \\ 6+24 & 12+32 \end{bmatrix} = \begin{bmatrix} 14 & 20 \\ 30 & 44 \end{bmatrix}$$

Division :
$$\begin{bmatrix} 14 & 20 \\ 30 & 44 \end{bmatrix} \div \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad ?$$

Inverses of multiplication = division

- One way to think about division in real numbers is multiplication by an inverse. Can we do something similar for matrices?

Ex. $15 \div 3 = 15 \cdot 3^{-1} = 15 \cdot \frac{1}{3} = 5$

Ex. $$\begin{bmatrix} 14 & 20 \\ 30 & 44 \end{bmatrix} \begin{bmatrix} -1 & \frac{1}{2} \\ \frac{3}{4} & -\frac{1}{4} \end{bmatrix} = \begin{bmatrix} -14+15 & 7-5 \\ -30+33 & 15-11 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} -1 & \frac{1}{2} \\ \frac{3}{4} & -\frac{1}{4} \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}^{-1}$$

Multiplicative inverses for real numbers

- Let x be a real number. The *(multiplicative) inverse* of x is another real number $x^{-1} = \frac{1}{x}$ such that $xx^{-1} = x^{-1}x = 1$.

$1^{-1} = 1$ $\quad 2^{-1} = \frac{1}{2}$ $\quad \pi^{-1} = \frac{1}{\pi}$ $\quad$ 0^{-1} doesn't exist

- Reversal of multiplication: $x^{-1}(xy) = (x^{-1}x)y = 1 \cdot y = y$

$\frac{1}{2}(2 \cdot 3) = \left(\frac{1}{2} \cdot 2\right) \cdot 3 = 3$

$\parallel$

$\frac{1}{2} \cdot 6 = 3$

$y \cdot 0 = 0$

Cannot be reversed

Matrix inverses (for square matrices)

- Let A be a square matrix. The *(multiplicative) inverse* of A is a matrix A^{-1} with the property that $AA^{-1} = A^{-1}A = I$, where I is the identity matrix.
  - If A has an inverse, then it is *invertible* or *nonsingular*.
  - If A does not have an inverse, then it is *noninvertible* or *singular.*
  - Theorem: for a square matrix, if $AA^{-1} = I$, then $A^{-1}A = I$.

$$A^{-1}(AB) = (A^{-1}A)B = IB = B$$

$$(BA)A^{-1} = B(AA^{-1}) = BI = B$$

Ex

$$\begin{bmatrix} -1 & \frac{1}{2} \\ \frac{3}{4} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} -2+3 & -4+4 \\ \frac{3}{2} - \frac{3}{2} & 3-2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} \begin{bmatrix} -1 & \frac{1}{2} \\ \frac{3}{4} & -\frac{1}{4} \end{bmatrix}$$

Finding a matrix inverse

$$\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} \begin{bmatrix} x & z \\ y & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2x + 4y & 2z + 4w \\ 6x + 8y & 6z + 8w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{cases} 2x + 4y = 1 \\ 6x + 8y = 0 \end{cases} \qquad \begin{cases} 2z + 4w = 0 \\ 6z + 8w = 1 \end{cases}$$

$$\left[\begin{array}{cc|c} 2 & 4 & 1 \\ \underline{6} & \underline{8} & 0 \end{array}\right] \qquad \left[\begin{array}{cc|c} 2 & 4 & 0 \\ \underline{6} & \underline{8} & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 2 & 4 & 1 & 0 \\ 6 & 8 & 0 & 1 \end{array}\right]$$

Can combine both augmented systems

Finding a matrix inverse (cont.)

$$\left[\begin{array}{cc|cc} 2 & 4 & 1 & 0 \\ 6 & 8 & 0 & 1 \end{array}\right]$$

$$R_1 \leftarrow \tfrac{1}{2} R_1$$
$$R_2 \leftarrow \tfrac{1}{6} R_2$$

$$\left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 1 & \frac{4}{3} & 0 & \frac{1}{6} \end{array}\right]$$

$$R_2 \leftarrow R_1 - R_2$$

$$\left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 0 & \frac{2}{3} & \frac{1}{2} & -\frac{1}{6} \end{array}\right]$$

$$R_2 \leftarrow \tfrac{3}{2} \cdot R_2$$

$$\left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{3}{4} & -\frac{1}{4} \end{array}\right]$$

$$R_1 \leftarrow R_1 - 2R_2$$

$$\left[\begin{array}{cc|cc} 1 & 0 & -1 & \frac{1}{2} \\ 0 & 1 & \frac{3}{4} & -\frac{1}{4} \end{array}\right]$$

Be very careful doing 2 steps at once, because you can mess things up.

$$\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & \frac{1}{2} \\ \frac{3}{4} & -\frac{1}{4} \end{bmatrix}$$

Matrix inversion through Gauss-Jordan

- Let A be a square $n \times n$ matrix. If we can row reduce the augmented matrix $[A|I]$ to the form $[I|B]$, then $A^{-1} = B$. Otherwise, the matrix A does not have an inverse.

Ex.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{R_1 \leftarrow R_1 - R_3 \\ R_2 \leftarrow R_2 - R_3}} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftarrow R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{matrix inverse}}$$

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{array}\right] \xrightarrow{R_2 = R_2 - 2R_1} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right]$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$ has no inverse

Try it out

$0.15 = \frac{3}{20}$ $-\frac{1}{4}\cdot\frac{20}{3} = -\frac{5}{3}$

- Remember the Leslie matrix $L = \begin{bmatrix} 2 & 3 \\ 0.5 & 0.9 \end{bmatrix}$ from our rabbit population model. Find the multiplicative inverse of L.

Solve $\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 0.5 & 0.9 & 0 & 1 \end{array}\right]$

make this the identity

Step 1: $R_1 = \frac{1}{2} R_1$

Step 2: $R_2 = R_2 - \frac{1}{2} R_1$

Step 3: Multiply to get leading 1.

A: $\begin{bmatrix} -2 & -3 \\ -0.5 & -0.9 \end{bmatrix}$

B: $\begin{bmatrix} 3 & -10 \\ -\frac{5}{3} & \frac{20}{3} \end{bmatrix}$

C: $\begin{bmatrix} 2 & 0.5 \\ 0.9 & 1 \end{bmatrix}$

D: $\begin{bmatrix} 3 & -\frac{5}{3} \\ -10 & \frac{20}{3} \end{bmatrix}$

E: None

Check

$\begin{bmatrix} 2 & 3 \\ 0.5 & 0.9 \end{bmatrix}\begin{bmatrix} 3 & -10 \\ -\frac{5}{3} & \frac{20}{3} \end{bmatrix} = \begin{bmatrix} 6-5 & -20+20 \\ 1.5-\frac{4.5}{3} & -5+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 0.5 & 0.9 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \frac{1}{2}} \left[\begin{array}{cc|cc} 1 & 1.5 & 0.5 & 0 \\ 0.5 & 0.9 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 1.5 & 0.5 & 0 \\ 0 & 0.15 & -0.25 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 1.5 & 0.5 & 0 \\ 0 & 1 & -\frac{5}{3} & \frac{20}{3} \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 3 & -10 \\ 0 & 1 & -\frac{5}{3} & \frac{20}{3} \end{array}\right]$

$0 - \frac{3}{2}\cdot\frac{20}{3} = -10$

$R_1 \leftarrow R_1 - 1.5R_2$

$0.5 - \left(-\frac{5}{3}\cdot\frac{3}{2}\right) = \frac{1}{2} + \frac{5}{2} = 3$

Solving linear systems using inverses

- Suppose $Ax = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = b$, where x is an unknown vector. Then we can solve $Ax = b$ by multiplying both sides on the *left* with A^{-1} if it exists. $x = A^{-1}Ax = A^{-1}b$

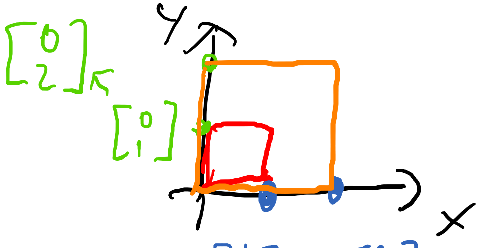
- Suppose you have a Leslie matrix $L = \begin{bmatrix} 2 & 3 \\ 0.5 & 0.9 \end{bmatrix}$ and a population vector $p_2 = \begin{bmatrix} 230 \\ 59 \end{bmatrix}$ in Year 2. What was the population vector p_1 in Year 1?

$$p_1 = \begin{bmatrix} 2 & 3 \\ 0.5 & 0.9 \end{bmatrix}^{-1} \begin{bmatrix} 230 \\ 59 \end{bmatrix} = \begin{bmatrix} 3 & -10 \\ -\frac{5}{3} & \frac{20}{3} \end{bmatrix} \begin{bmatrix} 230 \\ 59 \end{bmatrix} = \begin{bmatrix} 690 - 590 \\ -\frac{1150}{3} + \frac{1180}{3} \end{bmatrix} = \begin{bmatrix} 100 \\ 10 \end{bmatrix}$$

When does a matrix have an inverse?

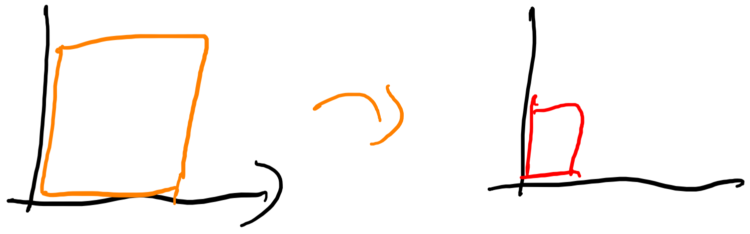
- Recall that matrices are transformations of vectors.

$$\begin{bmatrix}2&0\\0&2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}2x\\2y\end{bmatrix}$$



- A matrix has an inverse when you can reverse the transformation.

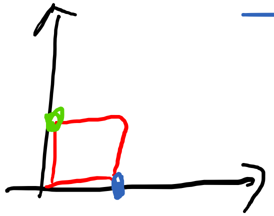
$$\begin{bmatrix}2&0\\0&2\end{bmatrix}^{-1}=\begin{bmatrix}\frac{1}{2}&0\\0&\frac{1}{2}\end{bmatrix}$$

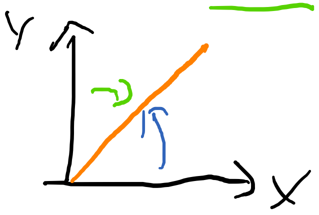


- But if a matrix sends two points to the same point, then you can't reverse that mapping.

$$\begin{bmatrix}0\\1\end{bmatrix}\to\begin{bmatrix}1\\1\end{bmatrix}\leftarrow\begin{bmatrix}1\\0\end{bmatrix}$$

$$\begin{bmatrix}1&1\\1&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x+y\\x+y\end{bmatrix}$$

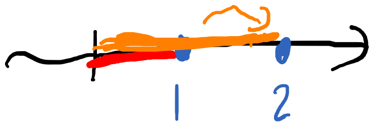




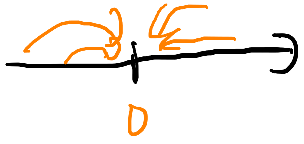
Matrices and length/area/volume scaling

- When a matrix squashes 1D line to a 0D point, that's irreversible.
  - Note that the length of a line gets scaled, but you get 0 length for a point.

$$\begin{bmatrix} 2 \end{bmatrix}\begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 2x \end{bmatrix}$$

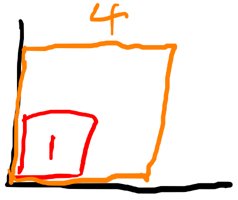


$$\begin{bmatrix} 0 \end{bmatrix}\begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

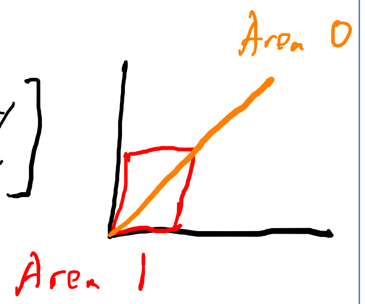


- When a matrix squashes a 2D square to a 1D line, that's irreversible.
  - Note that the area of a square gets scaled, but a line has area 0.

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

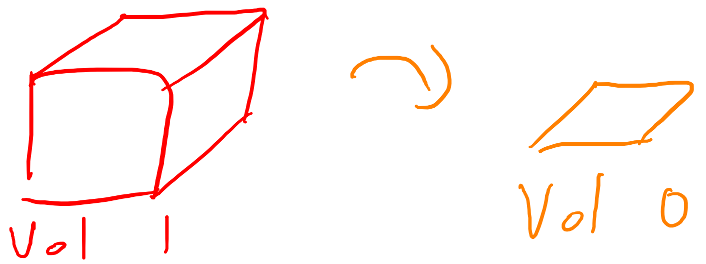


$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x+y \end{bmatrix}$$



- When a matrix squashes a 3D cube to a 2D plane, that's irreversible.
  - Note that a cube has nonzero volume, but a flat shape has volume 0.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$



Matrix Determinants

- The determinant of a 1×1 matrix $[a]$ is a.
- The determinant of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

  - Note that even though the notation | | looks like absolute values, determinants can be positive or negative.

Ex.

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1 \cdot 4 - 2 \cdot 3 = 4 - 6 = -2$$

$$\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 1 - 1 = 0$$

Try it out

- $\begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$ $= 0 \cdot 0 - (-1)(2) = 2$

A: 0
B: 1
C: 2
D: 3
E: None

- $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ $= 1 - (-1)(-1) = 0$

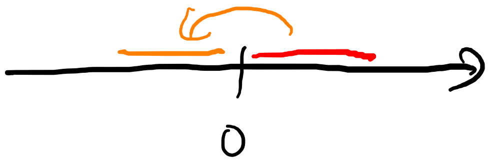
A: 0
B: 1
C: 2
D: 3
E: None

Determinants = (signed) scaling factor

1D

$[2][x] = [2x]$

$[-1][x] = [-x]$

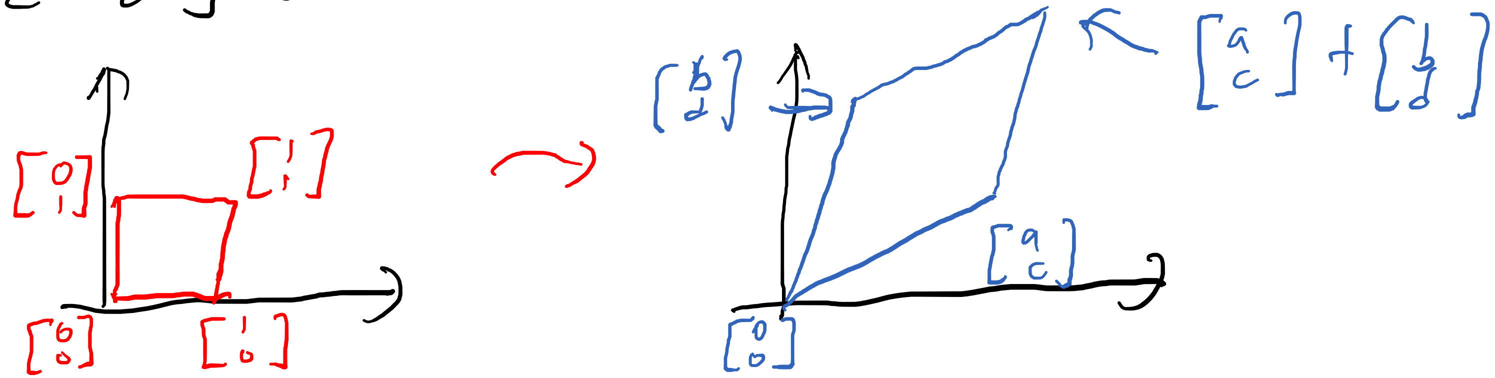


2D

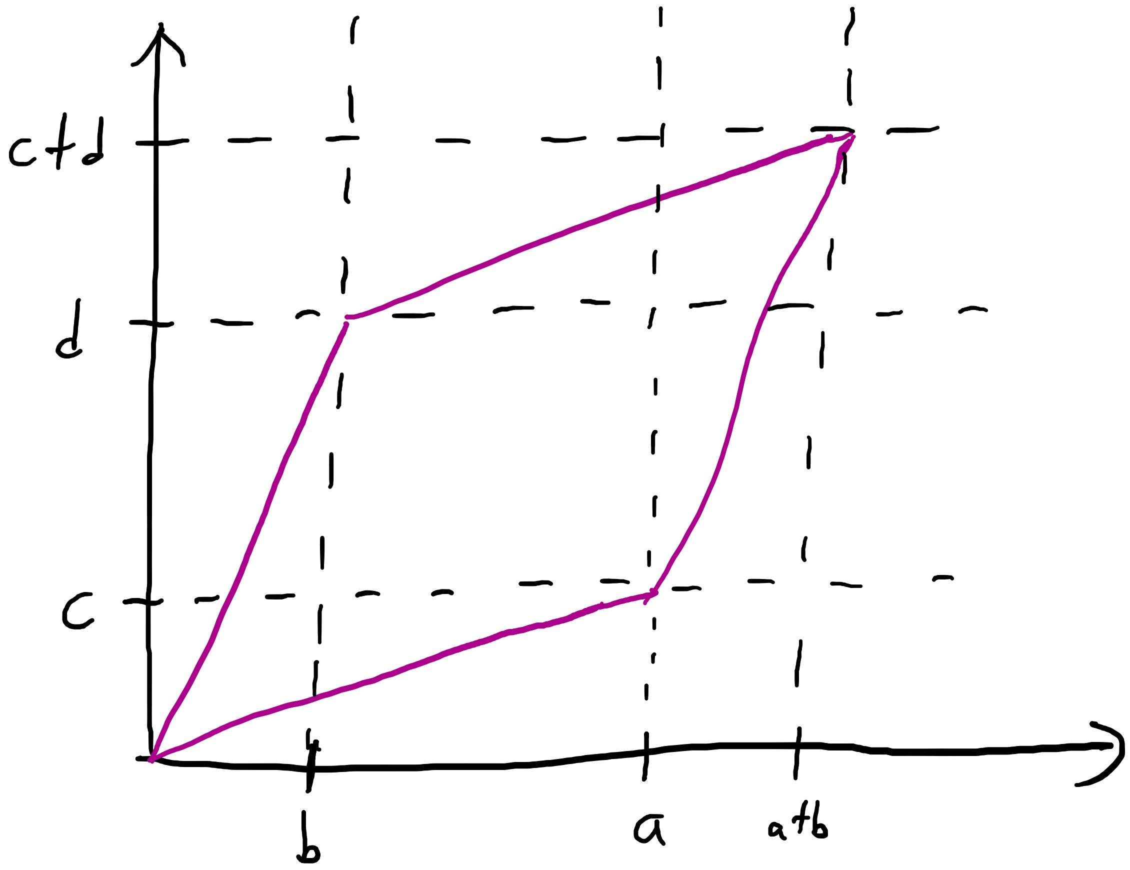
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

harder to interpret

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$$

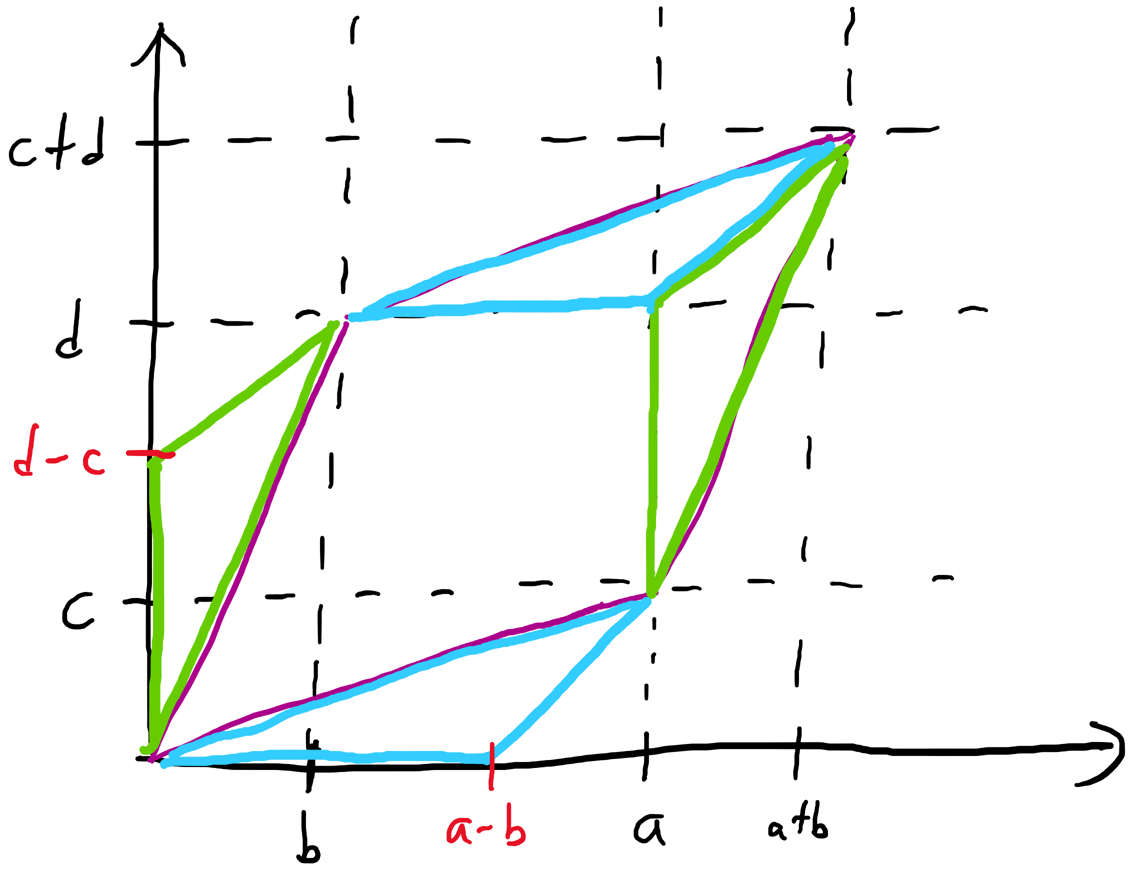


Area of parallelogram



Credit to John Wickerson, https://math.stackexchange.com/questions/29128/

Area of parallelogram



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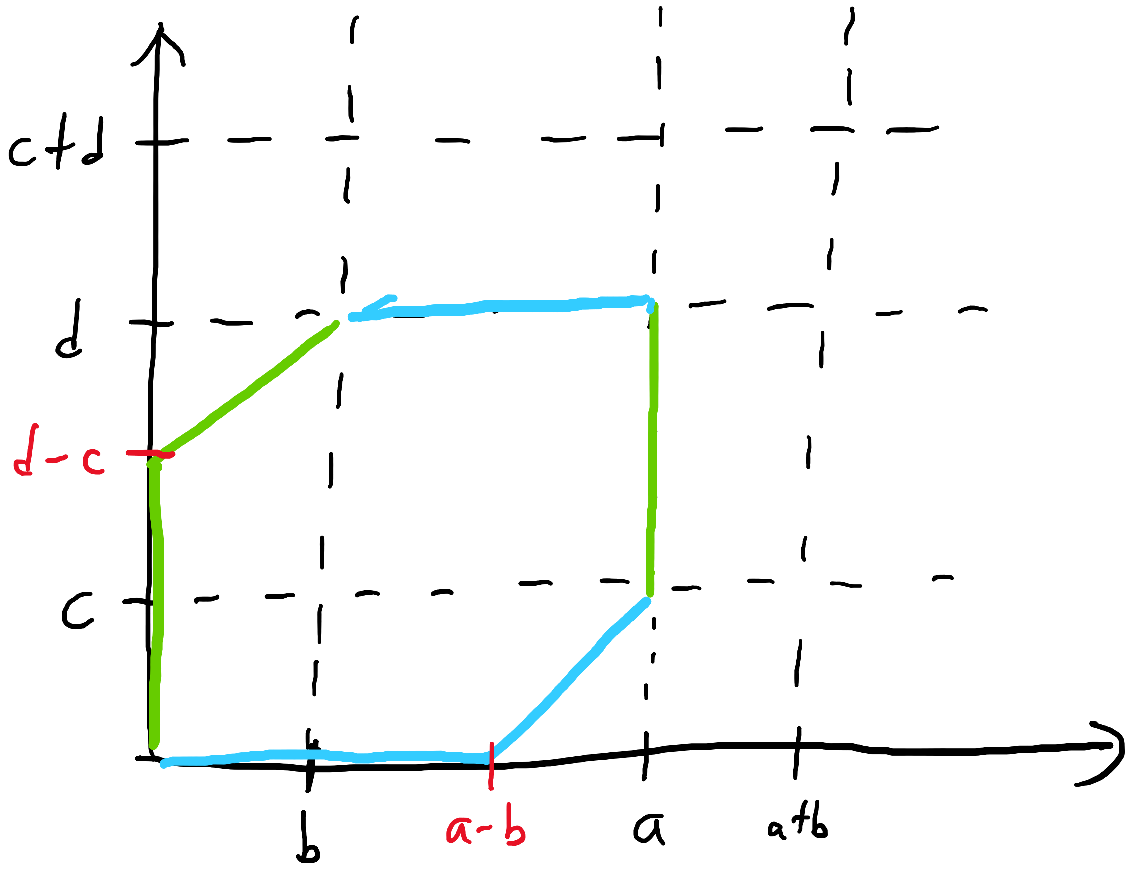
Area of parallelogram



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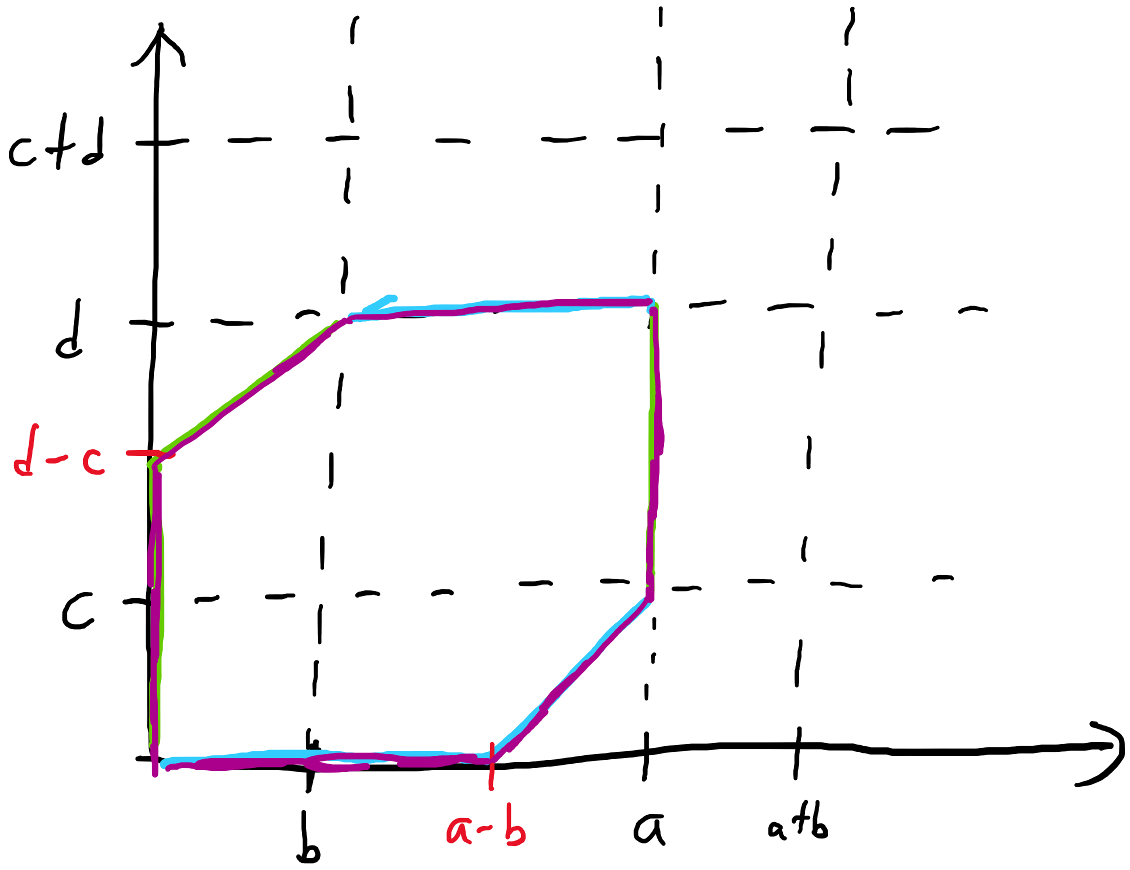
Area of parallelogram



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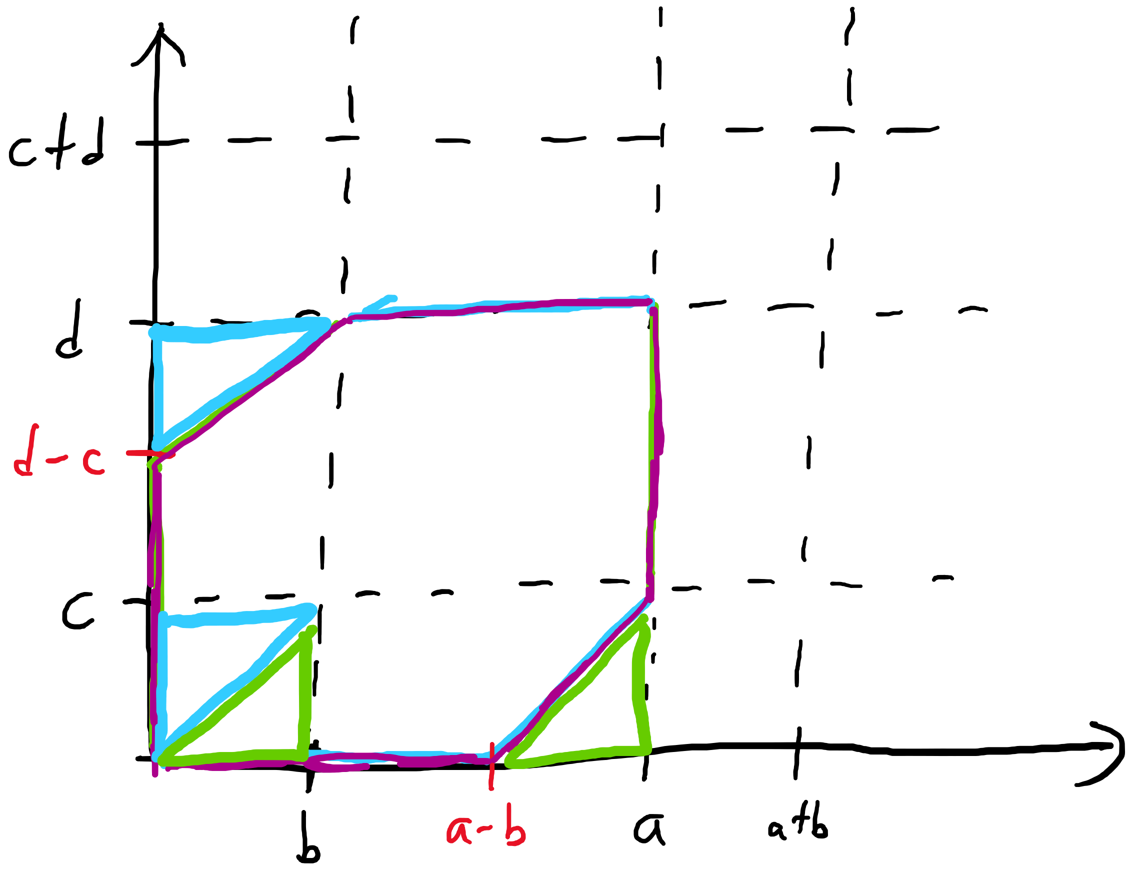
Area of parallelogram



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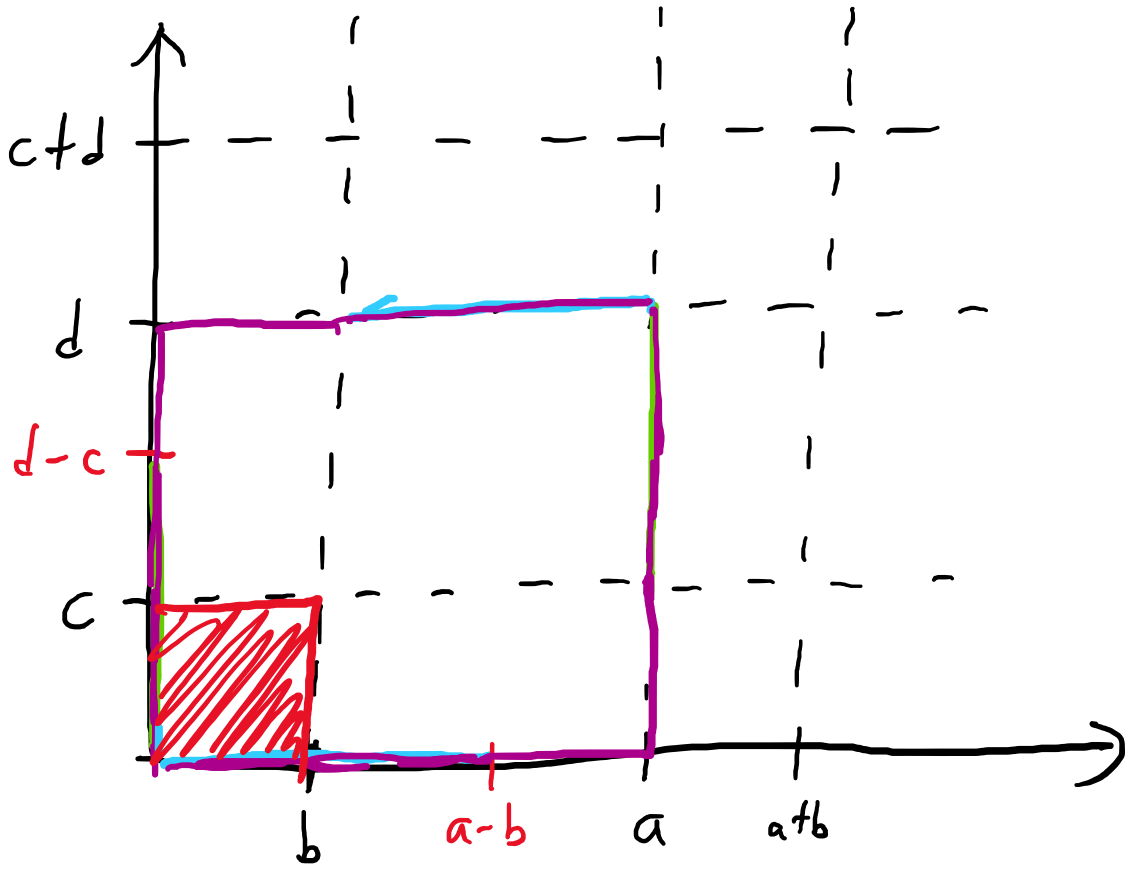
Area of parallelogram



Area =

$ad - bc$

Credit to John Wickerson, https://math.stackexchange.com/questions/29128/

Determinants and invertibility

- A square matrix is invertible if and only if its determinant is nonzero.
  - i.e. If a matrix squashes away a dimension, then it is not invertible, and vice versa.

$$\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 1 - 1 = 0 \quad \text{so not invertible}$$

- If A is a square matrix, and $Ax = 0$ for some vector $x \neq 0$, then $\det A = 0$.
  - i.e. If a matrix squashes some nonzero vector to zero, then it is not invertible.

$$x \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{so not invertible}$$

Determinants and matrix multiplication

- Since matrices are transformations, and determinants are a signed area, you can multiply together determinants:
- $\det(AB) = \det(A)\det(B)$, assuming A and B are square matrices of the same size.

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$$

$$\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \qquad \begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} = 4 \qquad \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} = 6-2 = 4$$

$$1 \cdot 4 = 4$$

Determinants, minors, and cofactors

- Let $A = [a_{ij}]$ be a square $n \times n$ matrix. Then we can define the ijth *minor* M_{ij} of A as the determinant of the matrix where you have removed the ith row and the jth column of A.
- The ijth cofactor C_{ij} of A is $C_{ij} = (-1)^{i+j} M_{ij}$.
- The determinant of A can be defined recursively by $|A| = a_{11}C_{11} + \cdots a_{1n}C_{1n}$ the sum of the entries in the first row and their respective cofactors.
    - (you can expand along any row or column using this formula)

3x3 determinant memory aid

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Example

A: 2
B: 5
C: 10
D: 32
E: None