More determinants; matrix eigenvalues and eigenvectors Lecture 4a – 2021-06-02

MAT A35 – Summer 2021 – UTSC

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Matrix Determinants

- The determinant of a 1×1 matrix $[a]$ is a.
- The determinant of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

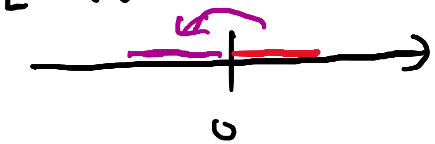
  - Note that even though the notation | | looks like absolute values, determinants can be positive or negative.

Determinants = (signed) scaling factor

1D:

$$\begin{bmatrix} 2 \end{bmatrix}\begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 2x \end{bmatrix}$$

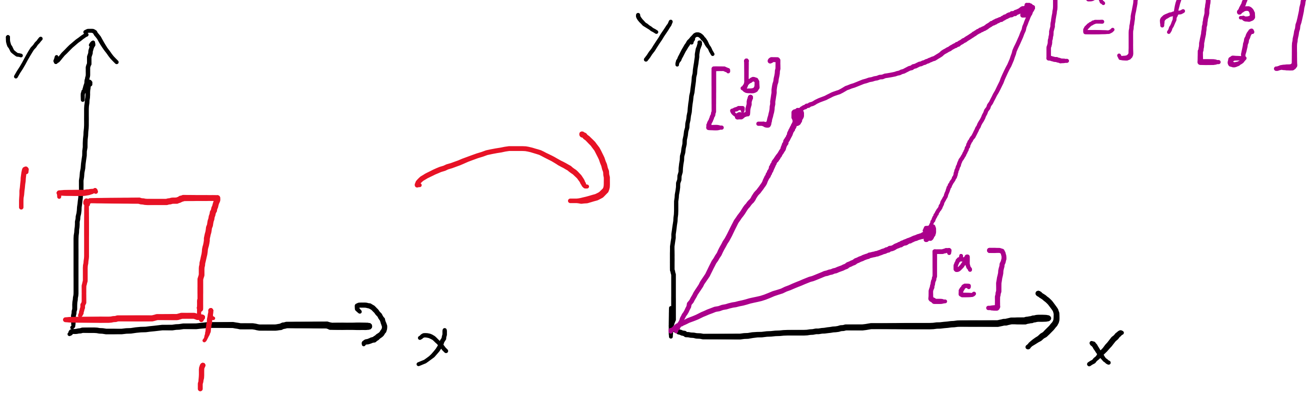
$$\begin{bmatrix} -1 \end{bmatrix}\begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} -x \end{bmatrix}$$



2D:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$ ← harder to interpret

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}, \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$$



Determinants, minors, and cofactors

- Let $A = [a_{ij}]$ be a square $n \times n$ matrix. Then we can define the ijth *minor* M_{ij} of A as the determinant of the matrix where you have removed the ith row and the jth column of A.
- The ijth cofactor C_{ij} of A is $C_{ij} = \underline{(-1)^{i+j}} M_{ij}$.
- The determinant of A can be defined recursively by $|A| = a_{11}C_{11} + \cdots a_{1n}C_{1n}$ the sum of the entries in the first row and their respective cofactors.
  - (you can expand along any row or column using this formula)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \quad M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, \quad M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$C_{11} = M_{11}$ $\quad C_{12} = -M_{12}$ $\quad C_{13} = M_{13}$

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$
$$- a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

3x3 determinant memory aid

$$\left[\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}\right]$$

− for upward diagonal

+ for downward diagonal

$$\left[\begin{vmatrix} a & b \\ c & d \end{vmatrix}\right]$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Example

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\begin{vmatrix} 0 & 1 & 0 \\ 2 & 1 & 3 \\ 6 & 5 & 0 \end{vmatrix} = 0 \cdot \begin{vmatrix} 1 & 3 \\ 5 & 0 \end{vmatrix} - 1 \begin{vmatrix} 2 & 3 \\ 6 & 0 \end{vmatrix} + 0 \begin{vmatrix} 2 & 1 \\ 6 & 5 \end{vmatrix}$$

$$= -1 \cdot (2 \cdot 0 - 18) = 18$$

$$= \left[\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 1 \\ 2 & 1 & 3 & 2 & 1 \\ 6 & 5 & 0 & 6 & 5 \end{array}\right] = 18$$

$$\begin{vmatrix} 0 & 1 & 0 & 1 \\ 2 & 1 & 3 & 2 \\ 6 & 5 & 0 & 3 \\ 0 & 0 & 0 & 2 \end{vmatrix} = -0 \cdot \begin{vmatrix} 1 & 0 & 1 \\ 1 & 3 & 2 \\ 5 & 0 & 3 \end{vmatrix} + 0 \begin{vmatrix} 0 & 0 & 1 \\ 2 & 3 & 2 \\ 6 & 0 & 3 \end{vmatrix} - 0 \begin{vmatrix} 0 & 1 & 1 \\ 2 & 1 & 2 \\ 6 & 5 & 3 \end{vmatrix} + 2 \begin{vmatrix} 0 & 1 & 0 \\ 2 & 1 & 3 \\ 6 & 5 & 0 \end{vmatrix}$$

$$= 2 \cdot 18 = 36$$

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

Try it out

- $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1 \cdot \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$

$= (45-48) - 2(36-42) + 3(32-35)$

$= -3 + 12 - 9 = 0$

$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow$ not invertible

$\rightarrow \det = 0$

A: -1
B: 0
C: 1
D: 2
E: None

- $\begin{vmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix} = 2 \begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{vmatrix}$

$= 2 \cdot 2 \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix}$

$= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$

A: 2
B: 5
C: 10
D: 32
E: None

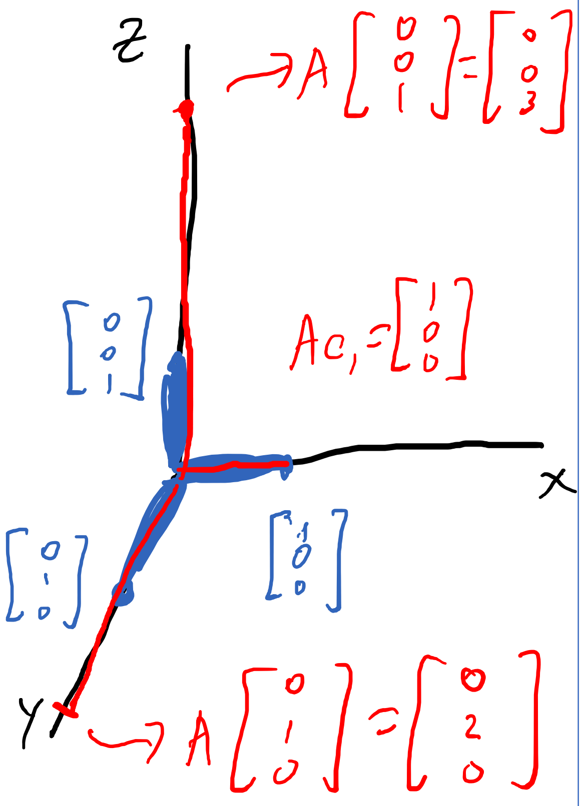
Diagonal matrices scale standard basis

- Scaling operations: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 2y \\ 3z \end{bmatrix}$

Basis vectors: $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$Ae_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = e_1$ $Ae_2 = 2e_2,\ Ae_3 = 3e_3$

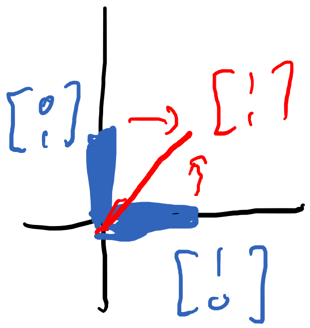
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

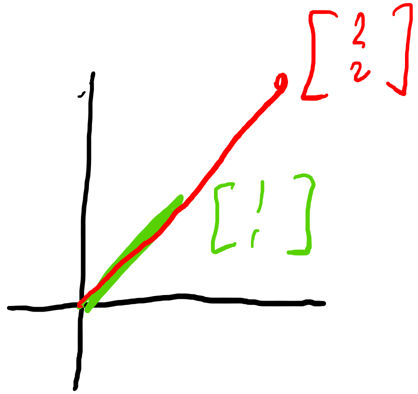


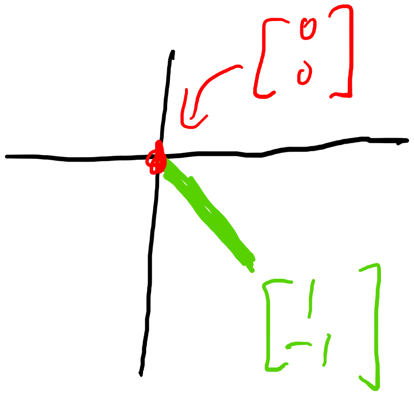
- An $n \times n$ diagonal matrix A with diagonal entries $\lambda_1, \ldots, \lambda_n$ scales the *standard basis* vectors $e_1, \ldots, e_n$, where e_i is a vector with 0's everywhere except a 1 in position i by $Ae_i = \lambda_i e_i$.

What about non-diagonal matrices?

- $\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}x+y\\ x+y\end{bmatrix}$







$$\underbrace{A\begin{bmatrix}0\\1\end{bmatrix} = A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}}_{\text{does not scale } \begin{bmatrix}0\\1\end{bmatrix} \text{ or } \begin{bmatrix}1\\0\end{bmatrix}}$$

$$\underbrace{A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}2\\2\end{bmatrix}}_{A \text{ scales } \begin{bmatrix}1\\1\end{bmatrix} \text{ by a factor of } 2} = 2\cdot\begin{bmatrix}1\\1\end{bmatrix}$$

$$A\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\underbrace{= 0\cdot\begin{bmatrix}1\\-1\end{bmatrix}}_{\text{scales } \begin{bmatrix}1\\-1\end{bmatrix} \text{ by a factor of } 0}$$

Eigenvalues and Eigenvectors

- Let A be an $n \times n$ square matrix, and let v be a non-zero vector of length n. Then if $Av = \lambda v$ for some number λ, then v is an eigenvector of A with corresponding eigenvalue λ. Together, they are also sometimes known as an eigenpair (λ, v).
  - An eigenvector v is a vector that gets scaled by a constant multiple λ (called an eigenvalue) when multiplied by A.
  - If v is an eigenvector for the eigenvalue λ, then so is kv, for any $k \neq 0$.

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ has two eigenpairs $\left(2, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$, $\left(0, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$

also works

$\left(2, \begin{bmatrix} 2 \\ 2 \end{bmatrix}\right)$ $\left(0, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$

$\left(2, \begin{bmatrix} -2 \\ -2 \end{bmatrix}\right)$ also works

$\left(0, \begin{bmatrix} -2 \\ 2 \end{bmatrix}\right)$

Try it out

- Let $A = \begin{bmatrix} -9 & 6 & 20 \\ 2 & 2 & -4 \\ -6 & 3 & 13 \end{bmatrix}$.
- Which of the following are eigenvectors of A?

A: $[2 \quad -6 \quad 5]^T$
B: $[6 \quad 1 \quad 3]^T$
C: $[12 \quad 2 \quad 6]^T$
D: All of the above
E: None of the above

$$\begin{bmatrix} -9 & 6 & 20 \\ 2 & 2 & -4 \\ -6 & 3 & 13 \end{bmatrix}\begin{bmatrix} 2 \\ -6 \\ 5 \end{bmatrix} = \begin{bmatrix} -18 - 36 + 100 \\ 4 - 12 - 20 \\ -12 - 18 + 65 \end{bmatrix} = \begin{bmatrix} 46 \\ -28 \\ 35 \end{bmatrix} \quad \times$$

$$\begin{bmatrix} -9 & 6 & 20 \\ 2 & 2 & -4 \\ -6 & 3 & 13 \end{bmatrix}\begin{bmatrix} 6 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -54 + 6 + 60 \\ 12 + 2 - 12 \\ -36 + 3 + 39 \end{bmatrix} = \underbrace{\begin{bmatrix} 12 \\ 2 \\ 6 \end{bmatrix} = 2 \cdot \begin{bmatrix} 6 \\ 1 \\ 3 \end{bmatrix}}_{} \quad \checkmark$$

C also eigenvector

Finding eigenvalues of a matrix

- Let A be a $n \times n$ matrix. If λ is an eigenvalue of A, then $\det(A - \lambda I) = 0$.

proof. $Av = \lambda v$ for some nonzero v.

$Av - \lambda v = 0$

$Av - \lambda I v = 0$ (because $v = Iv$)

$(A - \lambda I)v = 0$ (distributive prop.)

$\Rightarrow A - \lambda I$ is a singular/noninvertible matrix

$\Rightarrow \det(A - \lambda I) = 0$

Example

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

$$= \det\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right)$$

$$= \det\left(\begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} \right)$$

$$= (1-\lambda)^2 - 1 = 1 - 2\lambda + \lambda^2 - 1 = 0$$

$$\Rightarrow \quad \lambda^2 - 2\lambda = 0$$

$$\lambda(\lambda - 2) = 0$$

$$\Rightarrow \lambda = 0 \quad \text{or} \quad \lambda = 2$$

Try it out

- $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$\begin{vmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - \cancel{2\cdot 0} = 0$

$\Rightarrow \lambda = 1$

Find the eigenvalues:

A: 1
B: 2
C: 3
D: All of the above
E: None of the above

- $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$\begin{vmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{vmatrix} = 0$

$(1-\lambda)(4-\lambda) - 6 = 0$

$\lambda^2 - 5\lambda + 4 - 6 = 0$

$\lambda^2 - 5\lambda - 2 = 0$

$\lambda = \dfrac{5 \pm \sqrt{25+8}}{2}$

$\lambda = \frac{1}{2}(5 \pm \sqrt{33})$

Find the eigenvalues:

A: 1
B: 2
C: 3
D: All of the above
E: None of the above

$\lambda_1 = \frac{5}{2} + \frac{\sqrt{33}}{2}$

$\lambda_2 = \frac{5}{2} - \frac{\sqrt{33}}{2}$

Try it out

- $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

diagonal

triangular

triangular

Find the eigenvalues:
- A: 1
- B: 2
- C: 3
- D: All of the above
- E: None of the above

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} 2-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix}$$

$$= (1-\lambda)(2-\lambda)(3-\lambda) = 0$$

$$\Rightarrow \lambda = 1, 2, 3$$

$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \longrightarrow$ not diagonal

$\longrightarrow$ not triangular

- Triangular matrices have their eigenvalues on the diagonal.

Finding eigenvectors of a matrix

- $Av = \lambda v$, or alternately, $(A - \lambda I)v = 0$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0$$

$$(\lambda+1)(\lambda-1) = 0$$

$$\lambda = 1, -1 \qquad \Rightarrow \lambda_1 = 1, \lambda_2 = -1$$

$\lambda_1 = 1$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

Gaussian elim.

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$x - y = 0$$

$$\Rightarrow y = x$$

$$v_1 = \begin{bmatrix} x \\ x \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda_2 = -1$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$$

$$\begin{cases} y = -x \\ x = -y \end{cases}$$

$$x + y = 0$$
$$x + y = 0$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

$$v_2 = \begin{bmatrix} x \\ -x \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Example

$$A = \begin{bmatrix} 1 & 4 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$$

$\lambda_1 = 1$
$\lambda_2 = 2$
$\lambda_3 = 3$

$Av_1 = \lambda_1 v_1$
$Av_1 = \lambda_1 I v_1$
$Av_1 - \lambda_1 I v_1 = 0$
$(A - \lambda_1 I) v_1 = 0$

$\lambda_1 = 1$

$A v_1 = \lambda_1 v_1$, $v_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\begin{bmatrix} 1 & 4 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} 0 & 4 & 6 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 0 & 4 & 6 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$R_1 \leftarrow \frac{R_1}{4}$

$R_3 \leftarrow \frac{R_3}{2}$

$$\left[\begin{array}{ccc|c} 0 & 1 & \frac{3}{2} & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$R_1 = R_1 - \frac{3}{2} R_3$

$R_2 = R_2 - 5R_3$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow x = 0$
$y = 0$

$v_1 = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}$

$\downarrow$

$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Example (continued)

$\lambda_2 = 2$

$$\begin{bmatrix} 1 & 4 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}$$

$$\begin{cases} x + 4y + 6z = 2x \\ 2y + 5z = 2y \\ 3z = 2z \end{cases}$$

$\to z = 0$

$$\begin{cases} x + 4y = 2x \\ \cancel{2y = 2y} \\ z = 0 \end{cases} \to x = 4y$$

$$\to \begin{cases} x = 4y \\ z = 0 \end{cases}$$

$$v_2 = \begin{bmatrix} 4y \\ y \\ 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} -4 \\ -1 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 8 \\ 2 \\ 0 \end{bmatrix}$$

Try it out

- $A = \begin{bmatrix} 1 & 4 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$. What is the eigenvector corresponding to the eigenvalue $\lambda = 3$?

$$\left[\begin{array}{ccc|c} -2 & 4 & 6 & 0 \\ 0 & -1 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_1 \leftarrow R_1 + 2R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & -13 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x - 13z = 0 \\ y - 5z = 0 \end{cases} \Rightarrow \begin{matrix} x = 13z \\ y = 5z \end{matrix}$$

$$\begin{bmatrix} 13z \\ 5z \\ z \end{bmatrix} \xrightarrow{z=1} \begin{bmatrix} 13 \\ 5 \\ 1 \end{bmatrix}$$

Find the corresponding eigenvector:

A: $[0, 0, 1]^T$
B: $[0, 5, 1]^T$
C: $[13,5,1]^T$
D: All of the above
E: None of the above