More determinants; matrix eigenvalues and eigenvectors Lecture 4a – 2021-06-02

MAT A35 – Summer 2021 – UTSC

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Matrix Determinants

- The determinant of a 1×1 matrix $[a]$ is a.
- The determinant of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

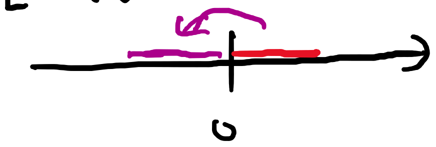
  - Note that even though the notation | | looks like absolute values, determinants can be positive or negative.

Determinants = (signed) scaling factor

1D:

$[2][x] = [2x]$

$[-1][x] = [-x]$

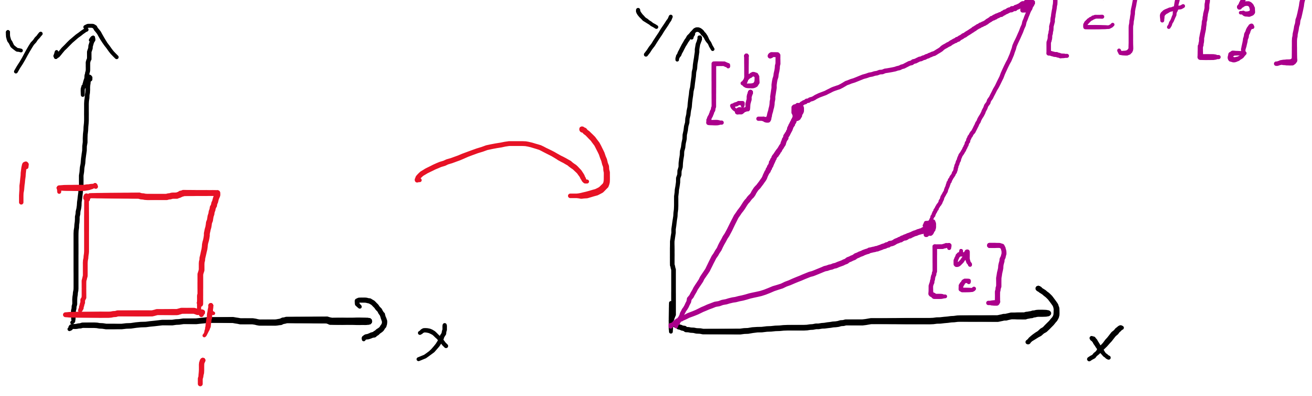


2D:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$$

← harder to interpret

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}, \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$$



Determinants, minors, and cofactors

- Let $A = [a_{ij}]$ be a square $n \times n$ matrix. Then we can define the ijth *minor* M_{ij} of A as the determinant of the matrix where you have removed the ith row and the jth column of A.
- The ijth cofactor C_{ij} of A is $C_{ij} = (-1)^{i+j} M_{ij}$.
- The determinant of A can be defined recursively by $|A| = a_{11}C_{11} + \cdots a_{1n}C_{1n}$ the sum of the entries in the first row and their respective cofactors.
  - (you can expand along any row or column using this formula)

3x3 determinant memory aid

Example

Try it out

- $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

A: -1
B: 0
C: 1
D: 2
E: None

- $\begin{vmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix}$

A: 2
B: 5
C: 10
D: 32
E: None

Diagonal matrices scale standard basis

- Scaling operations: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 2y \\ 3z \end{bmatrix}$

- An $n \times n$ diagonal matrix A with diagonal entries $\lambda_1, \dots, \lambda_n$ scales the *standard basis* vectors $e_1, \dots, e_n$, where e_i is a vector with 0's everywhere except a 1 in position i by $Ae_i = \lambda_i e_i$.

What about non-diagonal matrices?

- $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y \\ x + y \end{bmatrix}$

Eigenvalues and Eigenvectors

- Let A be an $n \times n$ square matrix, and let v be a non-zero vector of length n. Then if $Av = \lambda v$ for some number λ, then v is an eigenvector of A with corresponding eigenvalue λ. Together, they are also sometimes known as an eigenpair (λ, v).
  - An eigenvector v is a vector that gets scaled by a constant multiple λ (called an eigenvalue) when multiplied by A.
  - If v is an eigenvector for the eigenvalue λ, then so is kv, for any $k \neq 0$.

Try it out

- Let $A = \begin{bmatrix} -9 & 6 & 20 \\ 2 & 2 & -4 \\ -6 & 3 & 13 \end{bmatrix}$.
- Which of the following are eigenvectors of A?

A: $[2 \quad -6 \quad 5]^T$
B: $[6 \quad 1 \quad 3]^T$
C: $[12 \quad 2 \quad 6]^T$
D: All of the above
E: None of the above

Finding eigenvalues of a matrix

- Let A be a $n \times n$ matrix. If λ is an eigenvalue of A, then $\det(A - \lambda I) = 0$.

Example

Try it out

- $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

Find the eigenvalues:

A: 1
B: 2
C: 3
D: All of the above
E: None of the above

- $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Find the eigenvalues:

A: 1
B: 2
C: 3
D: All of the above
E: None of the above

Try it out

- $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Find the eigenvalues:

A: 1

B: 2

C: 3

D: All of the above

E: None of the above

- Triangular matrices have their eigenvalues on the diagonal.

Finding eigenvectors of a matrix

- $Av = \lambda v$, or alternately, $(A - \lambda I)v = 0$

Example

Example (continued)

Try it out

- $A = \begin{bmatrix} 1 & 4 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$. What is the eigenvector corresponding to the eigenvalue $\lambda = 3$?

Find the corresponding eigenvector:

A: $[0, 0, 1]^T$
B: $[0, 5, 1]^T$
C: $[13,5,1]^T$
D: All of the above
E: None of the above