Applications of eigenvalues and eigenvectors
Lecture 4b – 2021-06-02

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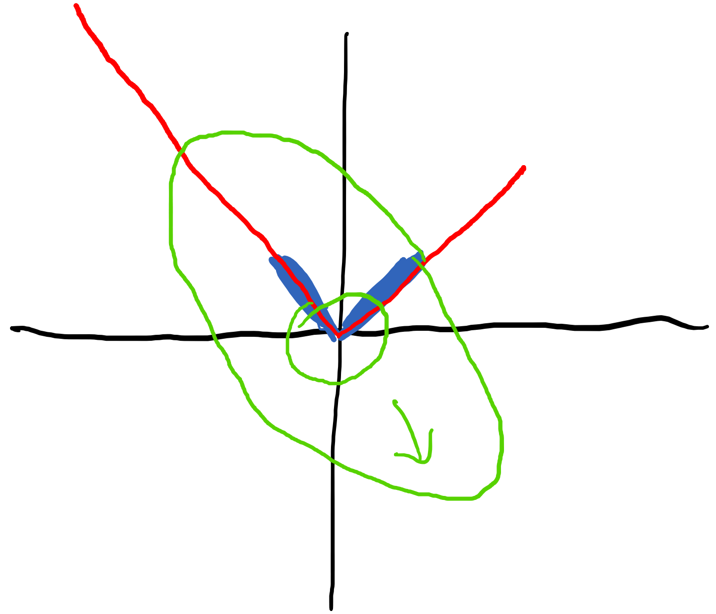
Interpreting eigenvectors and eigenvalues

- If we have n distinct eigenpairs of an $n \times n$ matrix A, we can interpret the "action" of A by what it does to the eigenvectors.

$$A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \qquad \lambda_1 = 2 \qquad v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 4 \qquad v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$



If $x = c_1 v_1 + c_2 v_2$

Then $Ax = c_1 A v_1 + c_2 A v_2$

$Ax = c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2$

Standard basis vectors of $\mathbb{R}^n$

- The standard basis of $\mathbb{R}^n$ is $e_1, \ldots, e_n$, where e_i is the vector with all 0's except a 1 in the ith entry.
- Any vector can be written as a *linear combination* of e_i's.

$\mathbb{R}^3$: $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix} = 5e_1 + 2e_2 - e_3$$

Other sets of basis vectors of $\mathbb{R}^n$

- A set of $v_1, \dots, v_n$ is a basis of $\mathbb{R}^n$ if every vector $w \in \mathbb{R}^n$ can be written as a linear combination $w = c_1 v_1 + \cdots + c_n v_n$.
- Any linearly independent set of n vectors in $\mathbb{R}^n$ is a basis of $\mathbb{R}^n$. A set of vectors is linearly independent if none of the vectors can be written as a linear combination of the other vectors.

Ex. $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^2$

Suppose $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = c \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Then $\cancel{1 = c = -1}$ contradiction

$\Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are linearly ind.

Note $\begin{bmatrix} x \\ y \end{bmatrix} = \frac{x+y}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{x-y}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\quad \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Eigenbasis of a square matrix

- If an $n \times n$ matrix A has n linearly independent eigenvectors, those eigenvectors form an eigenbasis.

$\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$ has an eigenbasis $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has only one linearly ind. eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so no eigenbasis

- Note that eigenvectors corresponding to different eigenvalues are necessarily linearly independent.

$\begin{cases} Av_1 = \lambda_1 v_1 \\ Av_2 = \lambda_2 v_2 \end{cases}$

Suppose $v_1 = c_1 v_2$

Then $Av_1 = c_1 Av_2$

$\Rightarrow \lambda_1 v_1 = c_1 \lambda_2 v_2$

$\Rightarrow \lambda_1 \underline{c_1 v_2} = \lambda_2 \underline{c_1 v_2}$.

$\Rightarrow \lambda_1 = \lambda_2$

- Also, can find all linearly independent eigenvectors corresponding to an eigenvalue by setting each of the free variables after Gaussian elimination.

$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix} \Rightarrow$

$2x = 2x$

$2y = 2y$

$z = 2z \Rightarrow z = 0$

2 free variables x & y

lin. ind.

$v = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Try it out: do the following have an eigenbasis?

- $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

Two distinct eigenvalues 1, 2 $\rightarrow$ Yes

$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

- $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

YES. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

$x = x$
$y = y$. 2 free variables

$v = \begin{bmatrix} x \\ y \end{bmatrix}$. $\rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$x=1, y=0$ $x=0, y=1$

- $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$

$\begin{cases} 2x + y = 2x \\ 2y = 2y \end{cases} \Rightarrow y = 0$

no constraint on x

$v = \begin{bmatrix} x \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

NO

- $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 1 = 0$

$\lambda^2 - 2\lambda = 0$

$\lambda(\lambda - 2) = 0$

$\lambda = 0, 2$

Two distinct eig. YES

A: Yes
B: No
C: Maybe
D: ???
E: None of the above

Population Growth Rates

- Suppose that the Leslie matrix G for a population has eigenvectors $v_1, \ldots, v_n$ with associated eigenvalues $\lambda_1, \ldots, \lambda_n$ respectively. If the initial population vector is $p = a_1 v_1 + \cdots + a_n v_n$, then the population after t time periods is

$$a_1 \lambda_1^t v_1 + \cdots + a_n \lambda_n^t v_n$$

proof. The pop vector after t time periods is $\boxed{G^t p}$

$= G^t [a_1 v_1 + \cdots + a_n v_n]$

$= a_1 G^t v_1 + \cdots + a_n G^t v_n$

$= a_1 \lambda_1^t v_1 + \cdots + a_n \lambda_n^t v_n$

$G v_i = \lambda_i v_i$

$G^t v_i = \lambda_i^t v_i$

Example

$$\begin{bmatrix}2\\1\end{bmatrix} = 1\cdot\begin{bmatrix}2\\1\end{bmatrix} + 0\cdot\begin{bmatrix}-1\\1\end{bmatrix}$$

- Consider an age-structured population model for birds where you have divided the group into young and old. Each old has only 1 hatchling each year, but survives with probability 1. Each young has 1.5 new hatchlings each year, but survives with only probability 0.5 to become old next year. If $p_0 = [2, 1]^T$, what is the population after 10 years?

$$p_0 = \begin{bmatrix}\text{young}\\ \text{old}\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix} \qquad L = \begin{bmatrix}1.5 & 1\\ 0.5 & 1\end{bmatrix}$$

$$\begin{vmatrix}1.5-\lambda & 1\\ 0.5 & 1-\lambda\end{vmatrix} = 1.5 - 2.5\lambda + \lambda^2 - 0.5 = 0$$

$$(\lambda-2)(\lambda-0.5)=0$$

$$\lambda = 2,\ 0.5$$

$$\lambda_1 = 2 \quad \begin{bmatrix}1.5 & 1\\ 0.5 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}2x\\2y\end{bmatrix} \Rightarrow 0.5x + y = 2y \Rightarrow x = 2y \qquad v_1 = \begin{bmatrix}2\\1\end{bmatrix}$$

$$\lambda_2 = 0.5 \quad \begin{bmatrix}1.5 & 1\\ 0.5 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0.5x\\0.5y\end{bmatrix} \Rightarrow x = -y \qquad v_2 = \begin{bmatrix}-1\\1\end{bmatrix}$$

$$p_{10} = 2^{10}\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}2048\\1024\end{bmatrix}$$

$$L^{10}\left(\begin{bmatrix}2\\1\end{bmatrix}\right)$$

$$L^{10}\left(\begin{bmatrix}2\\1\end{bmatrix} + 0\cdot\begin{bmatrix}-1\\1\end{bmatrix}\right)$$

$$= L^{10}\begin{bmatrix}2\\1\end{bmatrix} + L^{10}\cdot 0\cdot\begin{bmatrix}-1\\1\end{bmatrix}$$

$$= 2^{10}\begin{bmatrix}2\\1\end{bmatrix} + 0\cdot(0.5)^{10}\begin{bmatrix}-1\\1\end{bmatrix}$$

$$= 2^{10}\begin{bmatrix}2\\1\end{bmatrix}$$

Example

- What if the initial population were $p_0 = [3, 0]^T$?

$$\begin{bmatrix}3\\0\end{bmatrix} = c_1\begin{bmatrix}2\\1\end{bmatrix} + c_2\begin{bmatrix}-1\\1\end{bmatrix}$$

$$\Rightarrow c_1 = 1 \qquad c_2 = -1$$

$$\begin{bmatrix}3\\0\end{bmatrix} = 1\cdot\begin{bmatrix}2\\1\end{bmatrix} - 1\cdot\begin{bmatrix}-1\\1\end{bmatrix}$$

$$p_{10} = \underline{2^{10}\begin{bmatrix}2\\1\end{bmatrix}} - \underline{0.5^{10}\begin{bmatrix}-1\\1\end{bmatrix}} = \begin{bmatrix}2048\\1024\end{bmatrix} - \begin{bmatrix}-\frac{1}{1024}\\ \frac{1}{1024}\end{bmatrix}$$

$$\approx \begin{bmatrix}2048\\1024\end{bmatrix}$$

Try it out

- What if the initial population size was $p_0 = [0, 6]^T$? Which of the following answers is the closest to the population vector after 10 years?
- Recall $L = \begin{bmatrix} 1.5 & 1 \\ 0.5 & 1 \end{bmatrix}, \lambda_1 = 2, v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \lambda_2 = 0.5, v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

A: $[1000, 500]^T$
B: $[2000, 1000]^T$
C: $[4000, 2000]^T$
D: $[8000, 4000]^T$
E: $[16000, 8000]^T$

- Note that because exponentials grow super-fast, the long-term growth rate is dominated by the largest (magnitude) eigenvalue.

Try it out

- Consider a population with three life stages, newborn, juvenile, and adult, with the Leslie matrix $L = \begin{bmatrix} 0 & 6 & 8 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}$. At long time scales, what is the ratio of newborns to adults?

A: 2 to 1
B: 4 to 1
C: 8 to 1
D: 16 to 1
E: None

Advanced topic: complex eigenpairs

- Note that this will NOT be on Quiz 2.
- We’ve talked a lot about scaling by a constant multiple. But what happens if the numbers aren’t real?

- It turns out that imaginary eigenvalues correspond to rotations.
- Complex eigenvalues can be a combination of scaling and rotation.