Applications of eigenvalues and eigenvectors
Lecture 4b – 2021-06-02

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Interpreting eigenvectors and eigenvalues

- If we have n distinct eigenpairs of an $n \times n$ matrix A, we can interpret the “action” of A by what it does to the eigenvectors.

Standard basis vectors of $\mathbb{R}^n$

- The standard basis of $\mathbb{R}^n$ is $e_1, \ldots, e_n$, where e_i is the vector with all 0's except a 1 in the ith entry.
- Any vector can be written as a *linear combination* of e_i's.

Other sets of basis vectors of $\mathbb{R}^n$

- A set of $v_1, \dots, v_n$ is a basis of $\mathbb{R}^n$ if every vector $w \in \mathbb{R}^n$ can be written as a linear combination $w = c_1 v_1 + \cdots + c_n v_n$.
- Any linearly independent set of n vectors in $\mathbb{R}^n$ is a basis of $\mathbb{R}^n$. A set of vectors is linearly independent if none of the vectors can be written as a linear combination of the other vectors.

Eigenbasis of a square matrix

- If an $n \times n$ matrix A has n linearly independent eigenvectors, those eigenvectors form an eigenbasis.

- Note that eigenvectors corresponding to different eigenvalues are necessarily linearly independent.

- Also, can find all linearly independent eigenvectors corresponding to an eigenvalue by setting each of the free variables after Gaussian elimination.

Try it out: do the following have an eigenbasis?

- $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

- $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

- $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

- $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

A: Yes
B: No
C: Maybe
D: ???
E: None of the above

Population Growth Rates

- Suppose that the Leslie matrix G for a population has eigenvectors $v_1, \ldots, v_n$ with associated eigenvalues $\lambda_1, \ldots, \lambda_n$ respectively. If the initial population vector is $p = a_1 v_1 + \cdots + a_n v_n$, then the population after t time periods is
$$a_1 \lambda_1^t v_1 + \cdots + a_n \lambda_n^t v_n$$

Example

- Consider an age-structured population model for birds where you have divided the group into young and old. Each old has only 1 hatchling each year, but survives with probability 1. Each young has 1.5 new hatchlings each year, but survives with only probability 0.5 to become old next year. If $p_0 = [2, 1]^T$, what is the population after 10 years?

Example

- What if the initial population were $p_0 = [3, 0]^T$?

Try it out

- What if the initial population size was $p_0 = [0, 6]^T$? Which of the following answers is the closest to the population vector after 10 years?
- Recall $L = \begin{bmatrix} 1.5 & 1 \\ 0.5 & 1 \end{bmatrix}$, $\lambda_1 = 2$, $v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\lambda_2 = 0.5$, $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

A: $[1000, 500]^T$
B: $[2000, 1000]^T$
C: $[4000, 2000]^T$
D: $[8000, 4000]^T$
E: $[16000, 8000]^T$

- Note that because exponentials grow super-fast, the long-term growth rate is dominated by the largest (magnitude) eigenvalue.

Try it out

- Consider a population with three life stages, newborn, juvenile, and adult, with the Leslie matrix $L = \begin{bmatrix} 0 & 6 & 8 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}$. At long time scales, what is the ratio of newborns to adults?

A: 2 to 1
B: 4 to 1
C: 8 to 1
D: 16 to 1
E: None

Advanced topic: complex eigenpairs

- Note that this will NOT be on Quiz 2.
- We've talked a lot about scaling by a constant multiple. But what happens if the numbers aren't real?

- It turns out that imaginary eigenvalues correspond to rotations.
- Complex eigenvalues can be a combination of scaling and rotation.