Eigenbasis and applications Lecture 4c – 2021-06-04

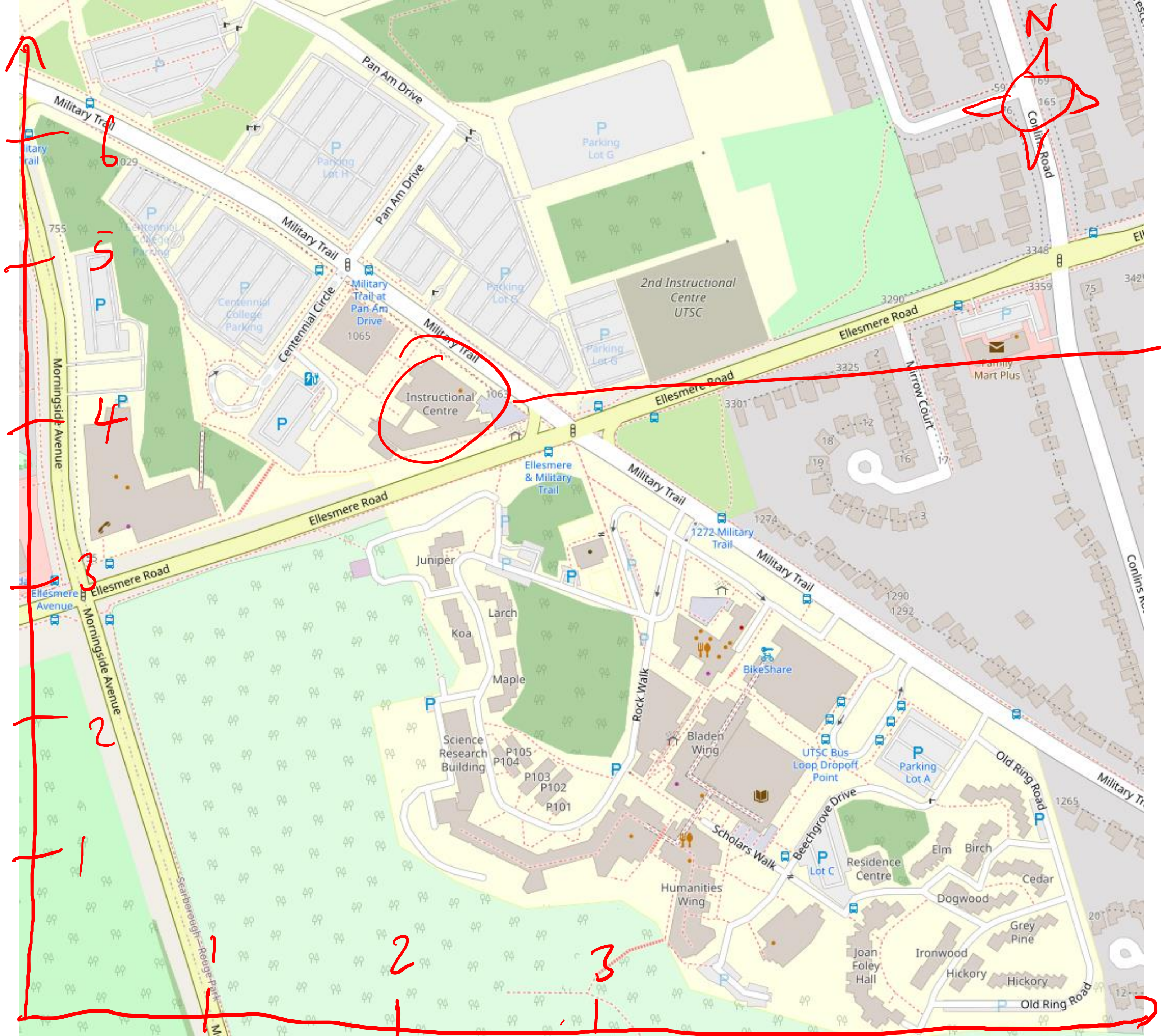
MAT A35 – Summer 2021 – UTSC

Prof. Yun William Yu

→ IC343

map grid C2

(3, 4)


N
Pan Am Drive
Military Trail
Parking Lot G
Parking Lot H
Centennial College Parking
Military Trail at Pan Am Drive
Centennial Circle
2nd Instructional Centre UTSC
Ellesmere Road
Instructional Centre
1065
1063
Family Mart Plus
Mirrow Court
Conlins Road
Morningside Avenue
Ellesmere & Military Trail
1272 Military Trail
Juniper
Larch
Koa
Maple
Rock Walk
BikeShare
Science Research Building
P105
P104
P103
P102
P101
Bladen Wing
UTSC Bus Loop Dropoff Point
Parking Lot A
Old Ring Road
Beechgrove Drive
Scholars Walk
Humanities Wing
Lot C
Residence Centre
Elm
Birch
Cedar
Dogwood
Grey Pine
Ironwood
Hickory
Joan Foley Hall
Ellesmere Avenue
Scarborough - Rouge Park
1
2
3
4
5
6

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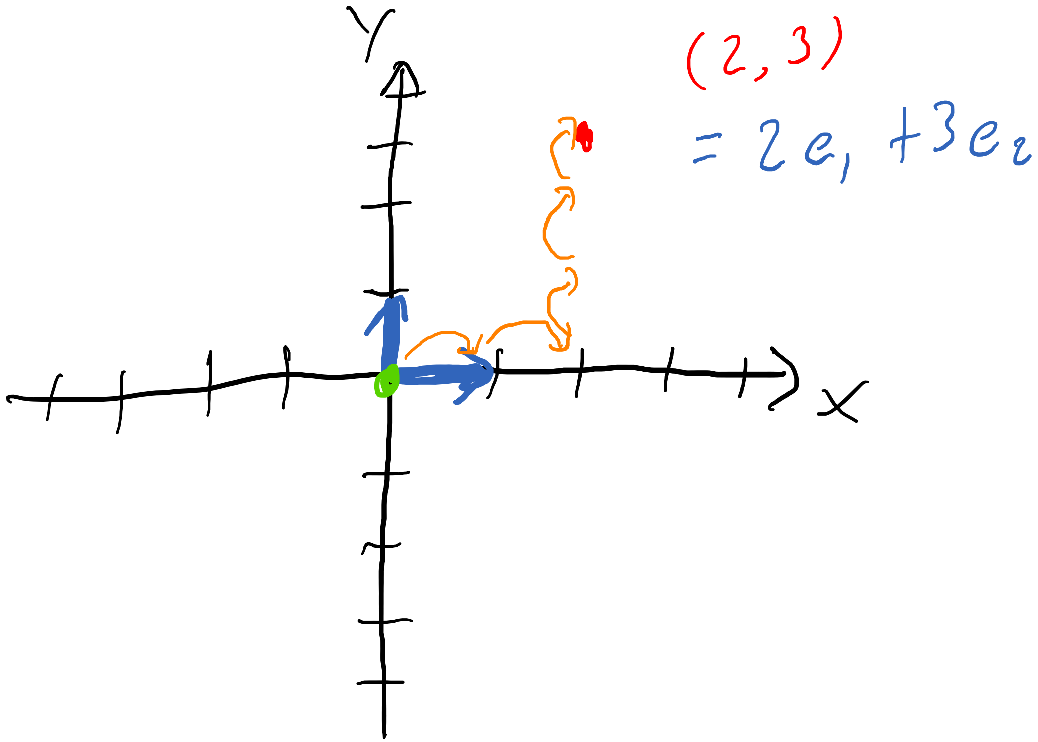
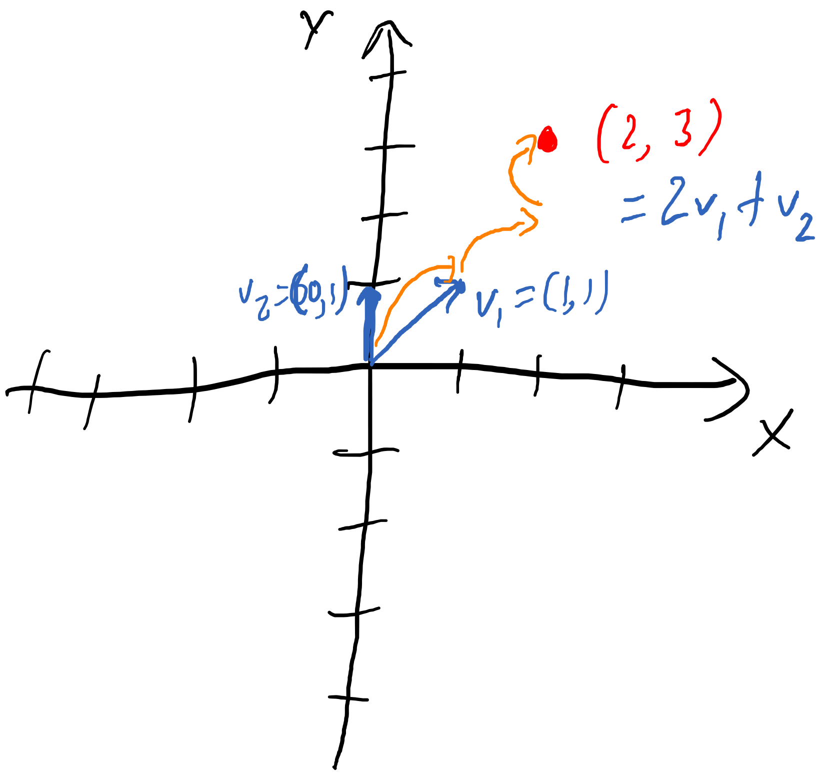
Specifying map coordinates

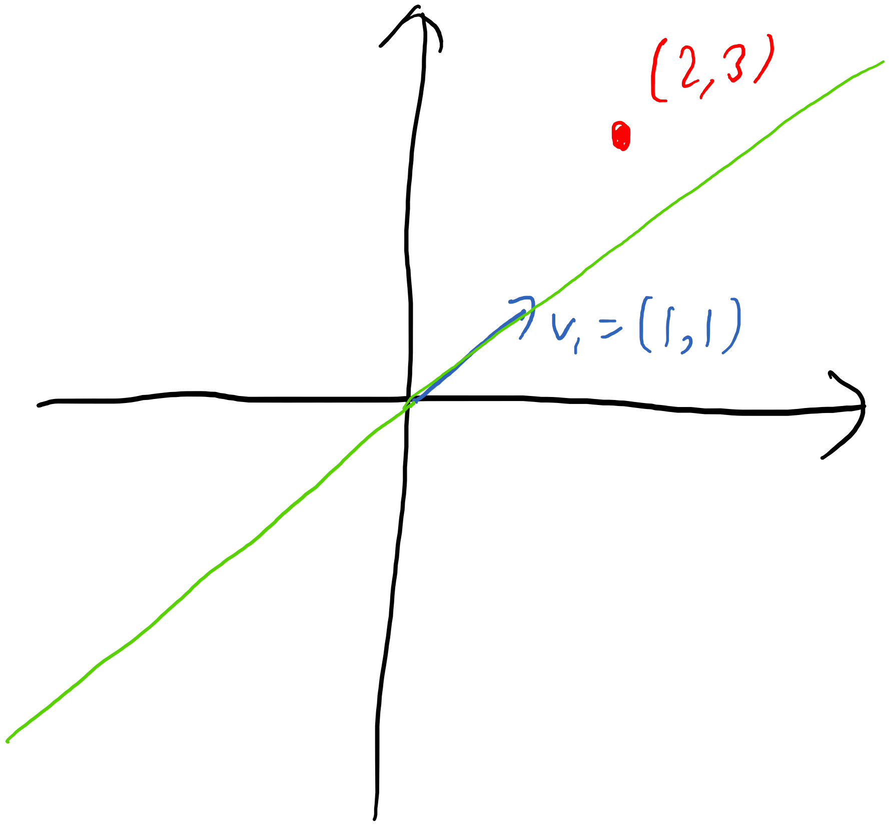
- On a 2-dimensional map, to specify a location, we need an origin point, two (independent) basic directions, and two coordinates.

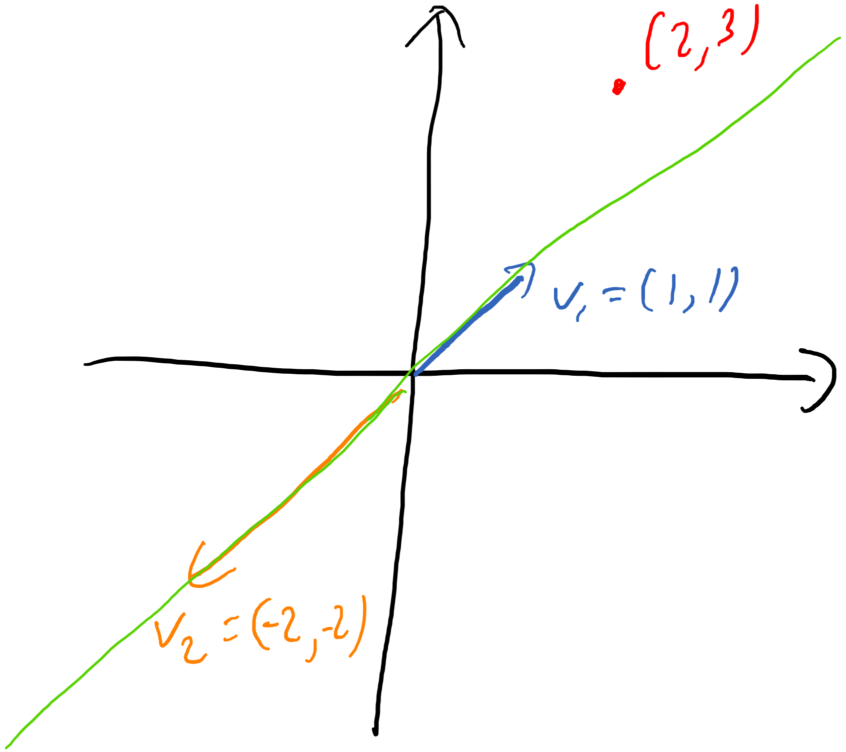




Independence of directions

- If we don't have enough basic directions (e.g. only one in 2D) or if the two basic directions chosen are not independent, then we cannot specify the location of any point (i.e. we do not have a basis)





Basis vectors of $\mathbb{R}^n$

$$\begin{bmatrix} 5 \\ 3 \end{bmatrix} \in \mathbb{R}^2 = 5 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- The standard basis of $\mathbb{R}^n$ is $e_1, \ldots, e_n$, where e_i is the vector with all 0's except a 1 in the ith entry.
  - Any vector can be written as a *linear combination* of e_i's.
- A set of $v_1, \ldots, v_n$ is a basis of $\mathbb{R}^n$ if every vector $w \in \mathbb{R}^n$ can be written as a linear combination $w = c_1 v_1 + \cdots + c_n v_n$.
- Any linearly independent set of n vectors in $\mathbb{R}^n$ is a basis of $\mathbb{R}^n$. A set of vectors is linearly independent if none of the vectors can be written as a linear combination of the other vectors.

  $v_1, \ldots, v_n$
  - Quick test for independence. Check that if $c_1 v_1 + \cdots + c_n v_n = 0$, then $c_1 = \cdots = c_n = 0$.

Test for linear independence

- The vectors $v_1, \ldots, v_n$ are linearly independent if $c_1 v_1 + \cdots c_n v_n = 0$ implies that $c_1 = \cdots = c_n = 0$.
  - Note: if you have n vectors in a space that is $< n$ dimensions, then they are not linearly independent.

Ex. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \end{bmatrix}$.

$$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$$

$$\begin{cases} c_1 + 2c_2 = 0 \\ c_1 - 2c_2 = 0 \end{cases}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & -2 & 0 \end{array}\right] \xrightarrow{R_2 \leftarrow R_1 - R_2} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 4 & 0 \end{array}\right] \xrightarrow{R_2 \leftarrow \frac{R_2}{4}} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow{R_1 \leftarrow R_1 - 2R_2} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

$$\begin{cases} c_1 = 0 \\ c_2 = 0 \end{cases} \Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \end{bmatrix} \text{ are independent}$$

Example

$$\begin{bmatrix}1\\2\\0\end{bmatrix},\ \begin{bmatrix}1\\0\\0\end{bmatrix},\ \begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$c_1\begin{bmatrix}1\\2\\0\end{bmatrix}+c_2\begin{bmatrix}1\\0\\0\end{bmatrix}+c_3\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}1&1&0\\2&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{cases}c_1+c_2=0\\2c_1+c_3=0\end{cases}$$

$$\begin{cases}c_2=-c_1\\c_3=-2c_1\end{cases}\Rightarrow$$

Sol $\begin{bmatrix}c_1\\-c_1\\-2c_1\end{bmatrix}\to\begin{bmatrix}1\\-1\\-2\end{bmatrix}$

$\Rightarrow$ not lin. ind.

Try it out: are the following a basis?

- $\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-2\\-2\end{bmatrix}$ NO. $\begin{bmatrix}-2\\-2\end{bmatrix} = -2\begin{bmatrix}1\\1\end{bmatrix}$
- $\begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}-5\\-4\end{bmatrix}$ YES. $\left[\begin{array}{cc|c}1 & -5 & 0\\3 & -4 & 0\end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - 3R_1} \left[\begin{array}{cc|c}1 & -5 & 0\\0 & 11 & 0\end{array}\right]$
- $\begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}-5\\-4\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$ NO. 3 vectors in 2D
- $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}$ NO. 2 vectors in 3D
- $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ $\left[\begin{array}{ccc|c}1 & 0 & 0 & 0\\2 & 1 & 0 & 0\\3 & 0 & 1 & 0\end{array}\right] \xrightarrow[R_3 \leftarrow R_3 - 3R_1]{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{ccc|c}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\end{array}\right]$ lin. ind. <u>YES</u>
- $\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}$ No. See previous slides. linearly dependent

A: Yes
B: No
C: Maybe
D: ???
E: None of the above

Matrices transform vectors

- The columns of a matrix A tell you where the matrix maps Ae_i to, where e_i are the standard basis vectors, but repeated application of A is nontrivial.

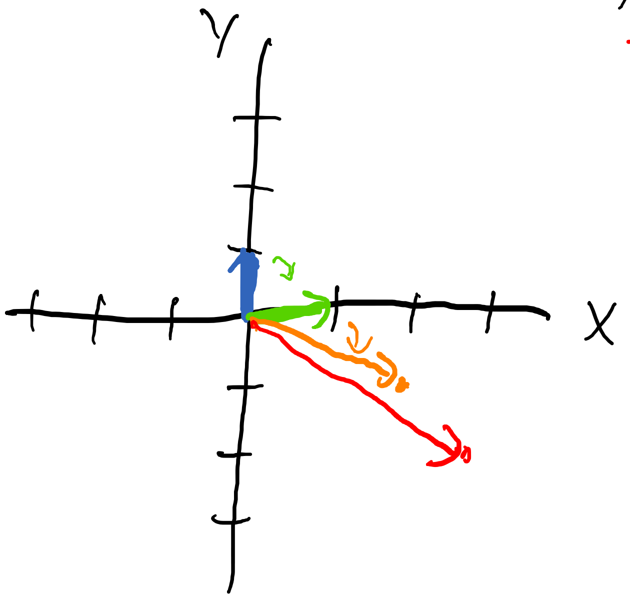
$$A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

$$Ae_1 = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} = A^2 e_2$$

$$Ae_2 = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} = A^3 e_2$$

$$A^{100} e_2 \qquad ?$$



$$e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$Ae_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A^2 e_2 = A(Ae_2) = A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = Ae_1$$

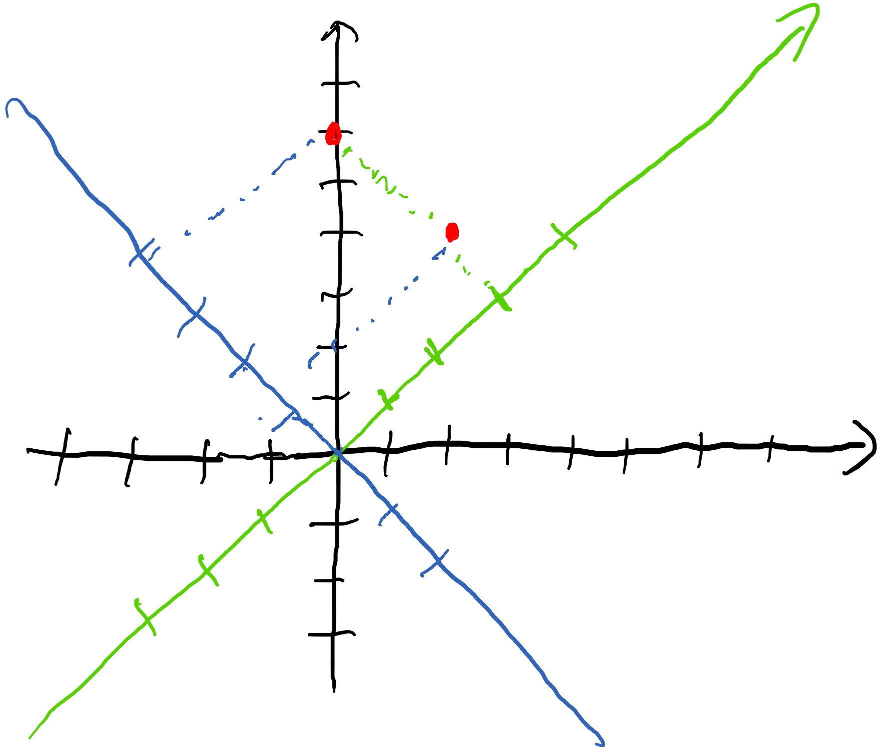
$$A^3 e_2 = A(A^2 e_2)$$

Interpreting eigenvectors and eigenvalues

- If we have n distinct eigenpairs of an $n \times n$ matrix A, we can interpret the "action" of A by what it does to the eigenvectors.

$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ has eigenpairs $\left(\boxed{3}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$

$\left(1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$



Note: $\begin{bmatrix} 2 \\ 4 \end{bmatrix} = 1 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + 3 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$A\begin{bmatrix} 2 \\ 4 \end{bmatrix} = 1 \cdot A\begin{bmatrix} -1 \\ 1 \end{bmatrix} + 3 \cdot A\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= 1 \cdot \left(\boxed{3}\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) + 3 \cdot \left(1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

$= \begin{bmatrix} 0 \\ 6 \end{bmatrix}$

Eigenbasis of a square matrix

- If an $n \times n$ matrix A has n linearly independent eigenvectors, those eigenvectors form an eigenbasis.
- Note that eigenvectors corresponding to different eigenvalues are necessarily linearly independent.
  - Thus, if an $n \times n$ matrix has n different eigenvalues, then it has an eigenbasis.
- You can solve for an eigenvector by setting $Ax = \lambda x$.
  - If λ is an eigenvalue, then you will have at least 1 free variable after solving, because any constant multiple of an eigenvector is an eigenvector.
  - If you have k free variables after solving, you can find k linearly independent eigenvectors for that *1* eigenvalue by setting each free variable to 1 and all the others to 0.

Try it out: do the following have an eigenbasis?

- $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ YES. Two diff eigenvalues

- $A = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$ NO. $\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$

  $2x = 2x$

  $x + 2y = 2y \Rightarrow x = 0$

  $\begin{bmatrix} 0 \\ y \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

- $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ YES $\begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) - 4 = 0$

  $4 - 5\lambda + \lambda^2 - 4 = 0$

  $\lambda^2 - 5\lambda = 0$

  $\lambda(\lambda - 5) = 0$

  $\lambda = 0, 5$

  Two eigenvals

- $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ YES.

  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are all eigenvectors

A: Yes
B: No
C: Maybe
D: ???
E: None of the above

Population Growth Rates

- Suppose that the Leslie matrix G for a population has eigenvectors $v_1, \ldots, v_n$ with associated eigenvalues $\lambda_1, \ldots, \lambda_n$ respectively. If the initial population vector is $p = a_1 v_1 + \cdots + a_n v_n$, then the population after t time periods is
$$a_1 \lambda_1^t v_1 + \cdots + a_n \lambda_n^t v_n$$

Example

- Consider an age-structured population model for birds where you have divided the group into young and old. Each old has only 1 hatchling each year, but survives with probability 1. Each young has 1.5 new hatchlings each year, but survives with only probability 0.5 to become old next year. If $p_0 = [6, 0]^T$, what is the population after 10 years?

$p_0 = \begin{bmatrix} \text{young} \\ \text{old} \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$

population vector

hatchlings from young

hatchlings from old

Leslie matrix

$\begin{bmatrix} 1.5 & 1 \\ 0.5 & 1 \end{bmatrix}$

survival of adults

survival of hatchlings → adults

Need to find $L^{10} p_0 = p_{10}$

Eigendecomposition of $L = \begin{bmatrix} 1.5 & 1 \\ 0.5 & 1 \end{bmatrix}$

Eigenvalues:

$$\begin{vmatrix} 1.5-\lambda & 1 \\ 0.5 & 1-\lambda \end{vmatrix} = 1.5 - 2.5\lambda + \lambda^2 - 0.5 = 0$$

$$\lambda^2 - 2.5\lambda + 1 = 0$$

$$(\lambda - 2)(\lambda - 0.5) = 0$$

$$\lambda_1 = 2 \qquad \lambda_2 = 0.5$$

Eigenvectors: $\lambda_1 = 2$

$$\begin{bmatrix} 1.5 & 1 \\ 0.5 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix} \longrightarrow \begin{matrix} 1.5x + y = 2x \\ 0.5x = y \end{matrix} \qquad \begin{bmatrix} x \\ 0.5x \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\lambda_2 = 0.5$

$$\begin{bmatrix} 1.5 & 1 \\ 0.5 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.5x \\ 0.5y \end{bmatrix} \longrightarrow \begin{matrix} x + y = 0 \\ x = -y \end{matrix} \qquad v_2 = \begin{bmatrix} -y \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Rewrite $\begin{bmatrix}6\\0\end{bmatrix} = c_1\begin{bmatrix}2\\1\end{bmatrix} + c_2\begin{bmatrix}-1\\1\end{bmatrix}$

$\left\{\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}$ eigenbasis

$\left\{ c_1\begin{bmatrix}2\\1\end{bmatrix} + c_2\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}6\\0\end{bmatrix} \right.$

$$\begin{bmatrix}2 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}6\\0\end{bmatrix}$$

$$\begin{array}{rl} & 2c_1 - c_2 = 6\\ +) & c_1 + c_2 = 0\\ \hline & 3c_1 = 6\end{array}$$

$c_1 = 2$

$c_2 = -2$

$$\boxed{\begin{bmatrix}6\\0\end{bmatrix} = \underline{2}\begin{bmatrix}2\\1\end{bmatrix} - \underline{2}\begin{bmatrix}-1\\1\end{bmatrix}}$$

Solve $\mathbf{p}_{10} = L^{10} p_0$ using eigenvectors

$$p_{10} = L^{10} \boxed{\begin{bmatrix} 6 \\ 0 \end{bmatrix}} = L^{10} \boxed{\left[2 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right]}$$

$L \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$L \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$= \underbrace{2 \cdot L^{10} \begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{} - \underbrace{2 L^{10} \begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{} = 2 \cdot 2^{10} \begin{bmatrix} 2 \\ 1 \end{bmatrix} - 2 \cdot \frac{1}{2^{10}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\lambda = 2$ $\qquad \lambda = \frac{1}{2}$

$$= 2^{11} \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \frac{1}{2^9} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4096 \\ 2048 \end{bmatrix} - \frac{1}{2^9} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \approx \begin{bmatrix} 4096 \\ 2048 \end{bmatrix}$$

Suppose

$\underline{Av = \lambda v}$. Then

$A^2 v = \lambda^2 v$

$A^3 v = A A^2 v$

$= AAAv$

$= \lambda^3 v$

$\underline{A^t v = \lambda^t v}$

$L^{10} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$A = L$

$t = 10$

$v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\lambda_1 = 2$

$L^{10} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$A = L$

$t = 10$

$v = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda_2 = \frac{1}{2}$

Try it out

- What if the initial population size was $p_0 = [0, 6]^T$? Which of the following answers is the closest to the population vector after 10 years?
- Recall $L = \begin{bmatrix} 1.5 & 1 \\ 0.5 & 1 \end{bmatrix}$, $\lambda_1 = 2$, $v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\lambda_2 = 0.5$, $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 0 \\ 6 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 4\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$P_{10} = 2 \cdot 2^{10}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 4 \cdot 0.5^{10}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \approx \begin{bmatrix} 4096 \\ 2048 \end{bmatrix}$$

A: $[1000, 500]^T$
B: $[2000, 1000]^T$
C: $[4000, 2000]^T$
D: $[8000, 4000]^T$
E: $[16000, 8000]^T$

- Note that because exponentials grow super-fast, the long-term growth rate is dominated by the largest eigenvalue.

Try it out

- Consider a population with three life stages, newborn, juvenile, and adult, with the Leslie matrix $L = \begin{bmatrix} 0 & 6 & 8 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}$. At long time scales, what is the ratio of newborns to adults?

$$\begin{vmatrix} -\lambda & 6 & 8 \\ 0.5 & -\lambda & 0 \\ 0 & 0.5 & -\lambda \end{vmatrix} = -\lambda^3 + 2 + 3\lambda = 0$$

$$\lambda^3 - 3\lambda - 2 = 0$$

$$\lambda^3 + \lambda^2 - (\lambda^2 + 3\lambda + 2) = 0$$

$$(\lambda+1)(\lambda^2 - \lambda - 2) = 0$$

$$(\lambda+1)(\lambda+1)(\lambda-2) = 0$$

$$\lambda = -1, \underline{2}$$

Solve for eigenvec corresponding to eigenval of 2.

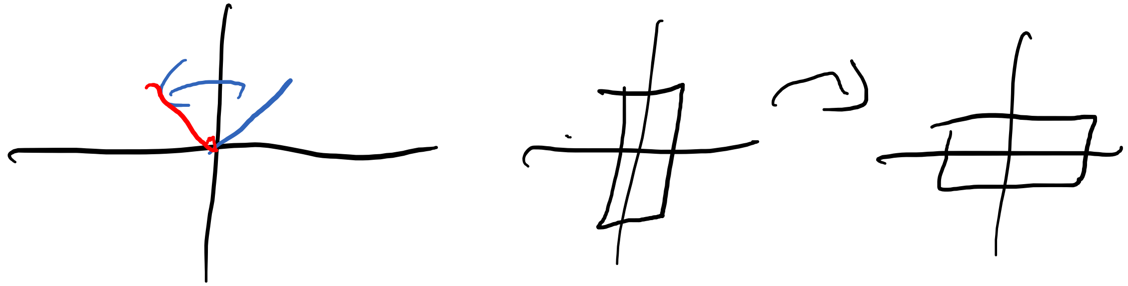
$$v = \begin{bmatrix} 16 \\ 4 \\ 1 \end{bmatrix}$$

A: 2 to 1
B: 4 to 1
C: 8 to 1
D: 16 to 1
E: None

Advanced topic: complex eigenpairs

- Note that this will NOT be on Quiz 2.
- We've talked a lot about scaling by a constant multiple. But what happens if the numbers aren't real?

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$



$$\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 \Rightarrow \lambda = \pm i$$

- It turns out that imaginary eigenvalues correspond to rotations.
- Complex eigenvalues can be a combination of scaling and rotation.