

Eigenbasis and applications

Lecture 4c – 2021-06-04

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Specifying map coordinates

- On a 2-dimensional map, to specify a location, we need an origin point, two (independent) basic directions, and two coordinates.

Independence of directions

- If we don't have enough basic directions (e.g. only one in 2D) or if the two basic directions chosen are not independent, then we cannot specify the location of any point (i.e. we do not have a basis)

Basis vectors of \mathbb{R}^n

- The standard basis of \mathbb{R}^n is e_1, \dots, e_n , where e_i is the vector with all 0's except a 1 in the i th entry.
 - Any vector can be written as a *linear combination* of e_i 's.
- A set of v_1, \dots, v_n is a basis of \mathbb{R}^n if every vector $w \in \mathbb{R}^n$ can be written as a linear combination $w = c_1 v_1 + \dots + c_n v_n$.
- Any linearly independent set of n vectors in \mathbb{R}^n is a basis of \mathbb{R}^n .
A set of vectors is linearly independent if none of the vectors can be written as a linear combination of the other vectors.
 - Quick test for independence. Check that if $c_1 v_1 + \dots + c_n v_n = 0$, then $c_1 = \dots = c_n = 0$.

Test for linear independence

- The vectors v_1, \dots, v_n are linearly independent if $c_1 v_1 + \dots + c_n v_n = 0$ implies that $c_1 = \dots = c_n = 0$.
 - Note: if you have n vectors in a space that is $< n$ dimensions, then they are not linearly independent.

Example

Try it out: are the following a basis?

• $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \end{bmatrix}$

• $\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ -4 \end{bmatrix}$

• $\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

• $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

• $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

• $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

A: Yes

B: No

C: Maybe

D: ???

E: None of the above

Matrices transform vectors

- The columns of a matrix A tell you where the matrix maps Ae_i to, where e_i are the standard basis vectors, but repeated application of A is nontrivial.

Interpreting eigenvectors and eigenvalues

- If we have n distinct eigenpairs of an $n \times n$ matrix A , we can interpret the “action” of A by what it does to the eigenvectors.

Eigenbasis of a square matrix

- If an $n \times n$ matrix A has n linearly independent eigenvectors, those eigenvectors form an eigenbasis.
- Note that eigenvectors corresponding to different eigenvalues are necessarily linearly independent.
 - Thus, if an $n \times n$ matrix has n different eigenvalues, then it has an eigenbasis.
- You can solve for an eigenvector by setting $Ax = \lambda x$.
 - If λ is an eigenvalue, then you will have at least 1 free variable after solving, because any constant multiple of an eigenvector is an eigenvector.
 - If you have k free variables after solving, you can find k linearly independent eigenvectors for that 1 eigenvalue by setting each free variable to 1 and all the others to 0.

Try it out: do the following have an eigenbasis?

• $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

• $A = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$

• $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

• $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

A: Yes

B: No

C: Maybe

D: ???

E: None of the above

Population Growth Rates

- Suppose that the Leslie matrix G for a population has eigenvectors v_1, \dots, v_n with associated eigenvalues $\lambda_1, \dots, \lambda_n$ respectively. If the initial population vector is $p = a_1 v_1 + \dots + a_n v_n$, then the population after t time periods is

$$a_1 \lambda_1^t v_1 + \dots + a_n \lambda_n^t v_n$$

Example



- Consider an age-structured population model for birds where you have divided the group into young and old. Each old has only 1 hatchling each year, but survives with probability 1. Each young has 1.5 new hatchlings each year, but survives with only probability 0.5 to become old next year. If $p_0 = [6, 0]^T$, what is the population after 10 years?

Eigendecomposition of $L = \begin{bmatrix} 1.5 & 1 \\ 0.5 & 1 \end{bmatrix}$

Rewrite $\begin{bmatrix} 6 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Solve $\mathbf{p}_{10} = L^{10}\mathbf{p}_0$ using eigenvectors

Try it out

- What if the initial population size was $p_0 = [0, 6]^T$? Which of the following answers is the closest to the population vector after 10 years?
- Recall $L = \begin{bmatrix} 1.5 & 1 \\ 0.5 & 1 \end{bmatrix}$, $\lambda_1 = 2$, $v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\lambda_2 = 0.5$, $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

- A: $[1000, 500]^T$
- B: $[2000, 1000]^T$
- C: $[4000, 2000]^T$
- D: $[8000, 4000]^T$
- E: $[16000, 8000]^T$

- Note that because exponentials grow super-fast, the long-term growth rate is dominated by the largest eigenvalue.

Try it out

- Consider a population with three life stages, newborn, juvenile, and adult, with the Leslie matrix $L = \begin{bmatrix} 0 & 6 & 8 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}$. At long time scales, what is the ratio of newborns to adults?

A: 2 to 1
B: 4 to 1
C: 8 to 1
D: 16 to 1
E: None

Advanced topic: complex eigenpairs

- Note that this will NOT be on Quiz 2.
- We've talked a lot about scaling by a constant multiple. But what happens if the numbers aren't real?

- It turns out that imaginary eigenvalues correspond to rotations.
- Complex eigenvalues can be a combination of scaling and rotation.