Functions of several variables
Lecture 5a – 2021-06-09

MAT A35 – Summer 2021 – UTSC

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What is a function?

- A function $f: V \rightarrow W$ takes an input in V and gives a (single) output in W.
- Easiest example is when $V = W = \mathbb{R}$, i.e. both are real numbers.

Ex. $f: \mathbb{R} \rightarrow \mathbb{R}$ where $f(x) = x^2$

$f(5) = 25$, $f(-1) = 1$, etc.

- Another classic example is when $V = W = \mathbb{C}$, both complex numbers.

Def. $i = \sqrt{-1}$

$\underbrace{a + bi, \quad a, b \in \mathbb{R}}_{= \mathbb{C}}$

Ex. $f: \mathbb{C} \rightarrow \mathbb{C}$ where $f(x) = x^2$

$f(-1) = 1$, $f(i) = i^2 = -1$, etc.

- We also have less "mathematical" examples. Let V be the set of days, and let W be the set of emotions, and let $f: V \rightarrow W$ be your dominant emotion on that day.

Ex. $f(\text{2021-June-9}) = \text{happy}$

$f(\text{2021-June-16}) = \text{anxious}$

Try it out: is this a function?

- f: [set of persons] $\rightarrow$ [set of colors], where given a person, f tells you what their favorite color is (assuming each person has exactly 1 favorite color)
- g: [set of persons] $\rightarrow$ [set of colors], where given a person, g tells you all the colors they like (can be multiple).
- h: [set of persons] $\rightarrow$ [set of all sets of colors], where given a person, h tells you all the colors they like (can be multiple).
- r: [photos on Reddit] $\rightarrow \{0,1\}$, where r returns 1 if the photo has a cat, and 0 if the photo does not have a cat.
- $r($  $) = 1,$ $r($  $) = 0$
- $f: \mathbb{R} \rightarrow \mathbb{R}$, where $f(x) = \pm\sqrt{x}$
- $g: \mathbb{R} \rightarrow \mathbb{R}$, where $g(x) = \sqrt{x}$
- $h: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, where $h(v) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} v$

A: Yes
B: No
C: Maybe
D: ???
E: None of the above

Functions of two variables

- Given $f: \mathbb{R}^2 \to \mathbb{R}$, we know that given any 2D vector $\begin{bmatrix} x \\ y \end{bmatrix}$, where $x, y \in \mathbb{R}$ are real numbers and the output $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ is another real number.
  - Often, we will write $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ as $f(x, y)$ for convenience, so f can be thought of as a function of two real variables.

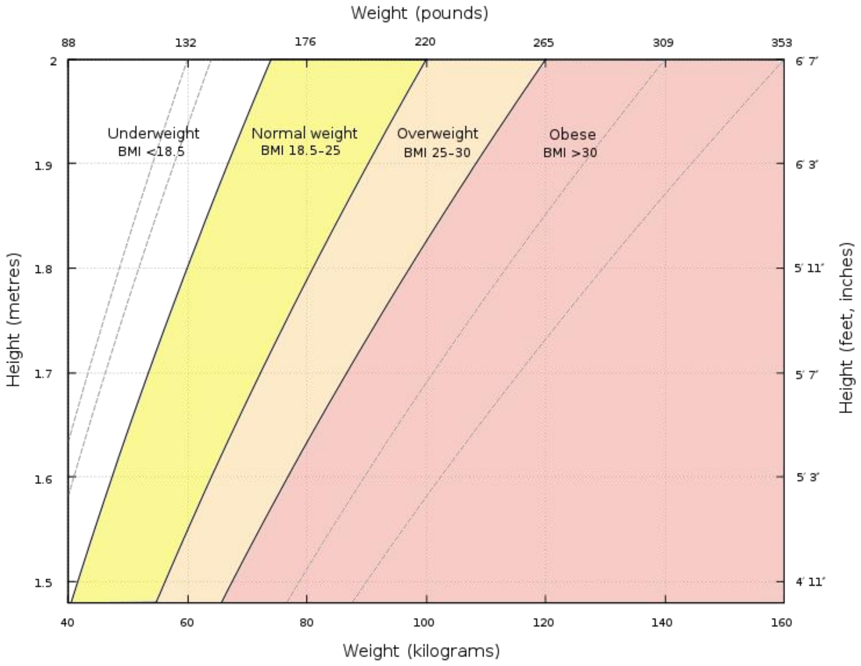
Ex. $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2x + y$ $\quad f\left(\begin{bmatrix} 1 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix} = 7$

Ex. $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x^2 + y^2$ $\quad f(1, 5) = 1 + 25 = 26$

Ex. $g(x, y) = x^2 e^y + 5xy$ $\quad g(1, 5) = e^5 + 25$

Application – body mass index (BMI)

- The Body-Mass Index (BMI) was developed by Adolphe Quetelet to (approximately) quantify obesity.
- $B(m,h) = \frac{m}{h^2}$, where m is mass in kilograms and h is height in meters.



https://commons.wikimedia.org/wiki/File:BMI_chart.svg

Ex

$m = 68$ kg

$h = 1.71$ meters

$$B(m,h) = \frac{68\text{ kg}}{(1.71\text{ m})^2} = 23.3\ \text{kg}/\text{m}^2$$

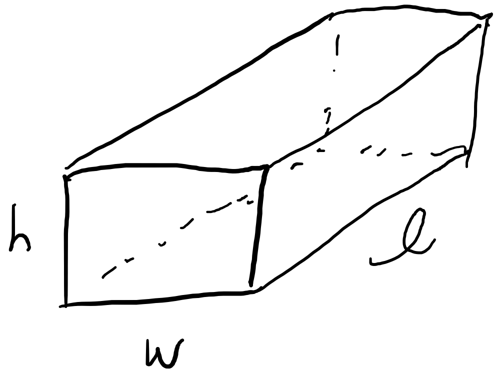
$\rightarrow$ "normal weight"

When was BMI invented?
A: Before 1700 A.D.
B: 1700-1800 A.D.
C: 1800-1900 A.D.
D: 1900-2000 A.D.
E: After 2000 A.D.

Functions of several variables

- $f: \mathbb{R}^n \to \mathbb{R}$ is a function from n real numbers that outputs 1 real number.

Ex. Volume of a right rectangular prism



$V(h, w, l) = hwl$

If $h = 1$ cm

$w = 1.5$ cm

$l = 4$ cm

$V(h, w, l) = 6 \text{ cm}^3$

Ex. $f(x, y, z) = 0$ is a multivariable constant function.

Geometric interpretation of functions

- $f: \mathbb{R} \to \mathbb{R}$ "maps" a real number to a real number.
- We can think of the function as pairs of numbers in $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$, which is what we do when we draw the graph of a function.

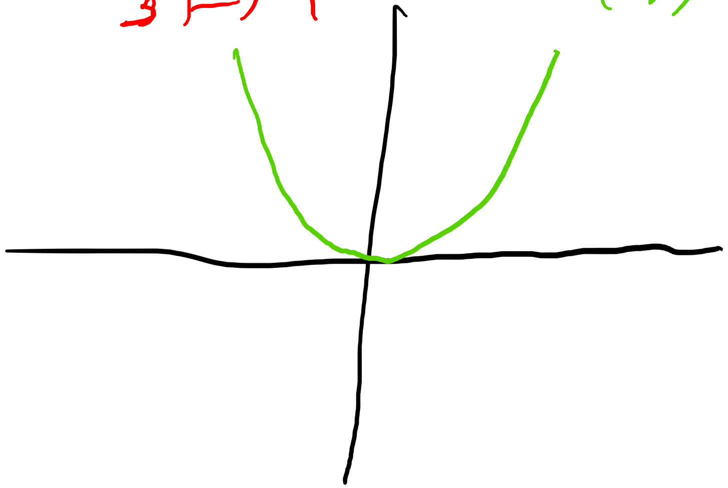
Ex. $f(x) = x^2$

$f(1) = 1$ $\quad 1 \mapsto 1$ $\quad (1, 1)$

$f(2) = 4$ $\quad 2 \mapsto 4$ $\quad (2, 4)$

$f(3) = 9$ $\quad 3 \mapsto 9$ $\quad (3, 9)$

$x \qquad y = f(x) = x^2$

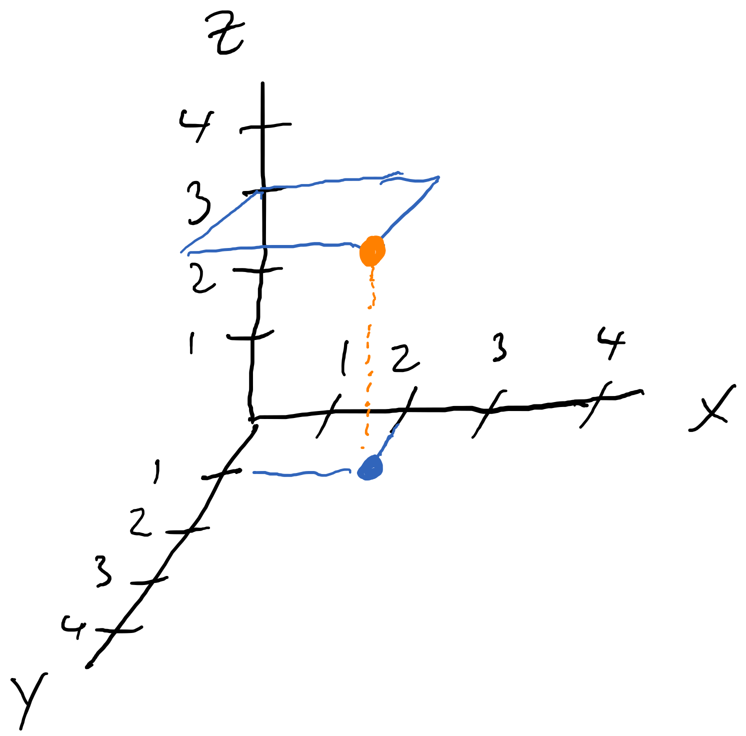


Geom. interpret. of 2-variable function

- $f: \mathbb{R}^2 \to \mathbb{R}$ “maps” a pair of numbers to a single number.
- We can think of the function as a list of ordered pairs $\big((x, y), z\big) \in \mathbb{R}^2 \times \mathbb{R}$, where the first part of the ordered pair is a pair $(x, y) \in \mathbb{R}^2$ itself. We can then “graph” the function by drawing a surface in $\mathbb{R}^3$.

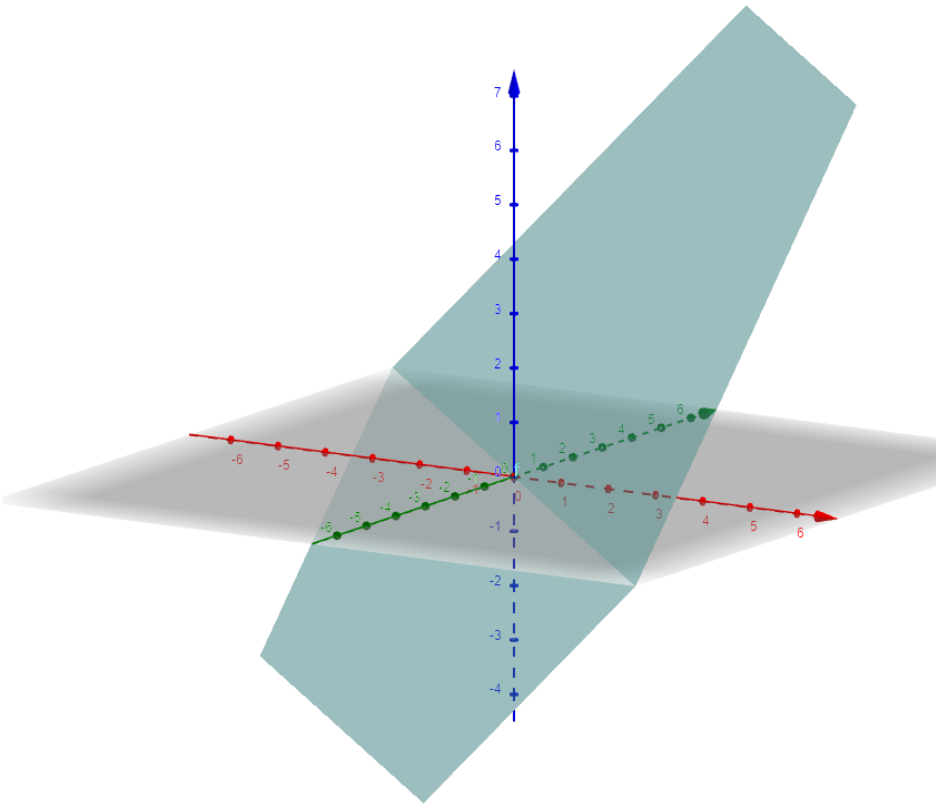
Ex. $f(x,y) = x+y$

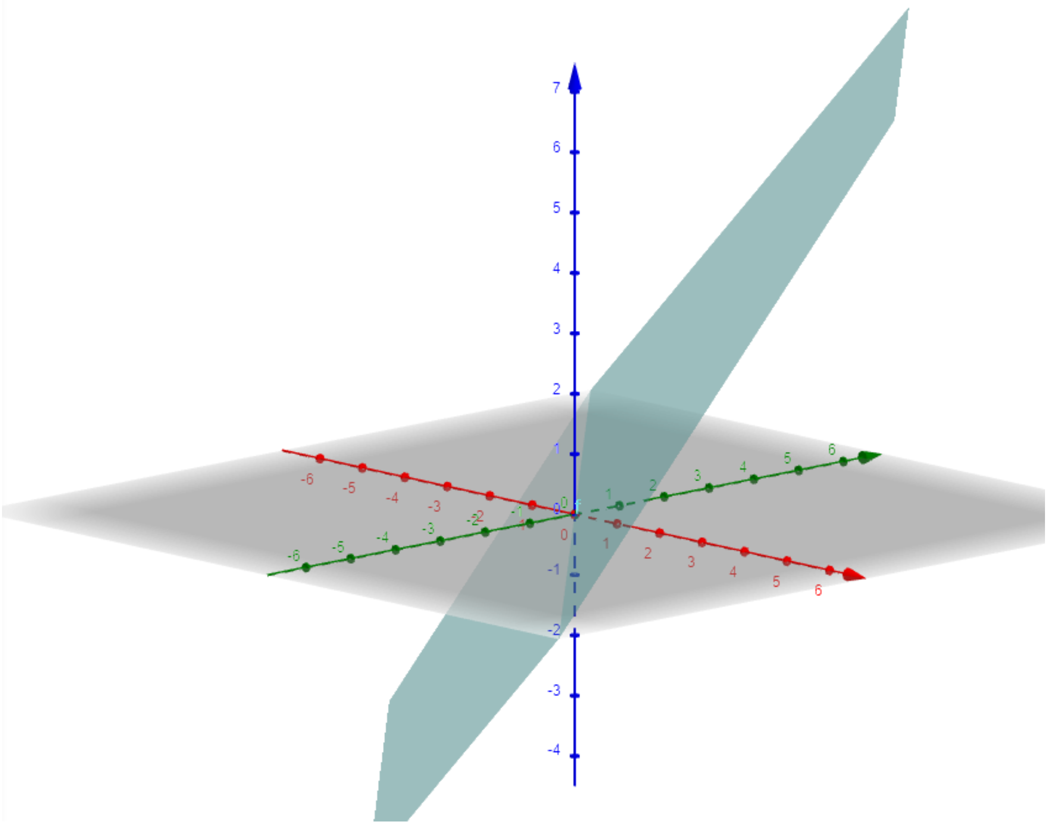
$f(0,1)=1$	$(0,1) \mapsto 1$	$(0,1,1)$
$f(0,2)=2$	$(0,2) \mapsto 2$	$(0,2,2)$
$f(1,1)=2$	$(1,1) \mapsto 2$	$(1,1,2)$
$f(2,1)=3$	$(2,1) \mapsto 3$	$(2,1,3)$



3D plotting

- https://www.geogebra.org/3d?lang=en
  - $z = 0$
  - $f(x, y) = x + y$





More 3D plotting examples

- https://www.geogebra.org/3d?lang=en
  - $z = 0$
  - $f(x, y) = \frac{\sin(x^2+y^2)}{x^2+y^2}$

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f(x,y)=sin(x^2+y^2)/(x^2+y^2)
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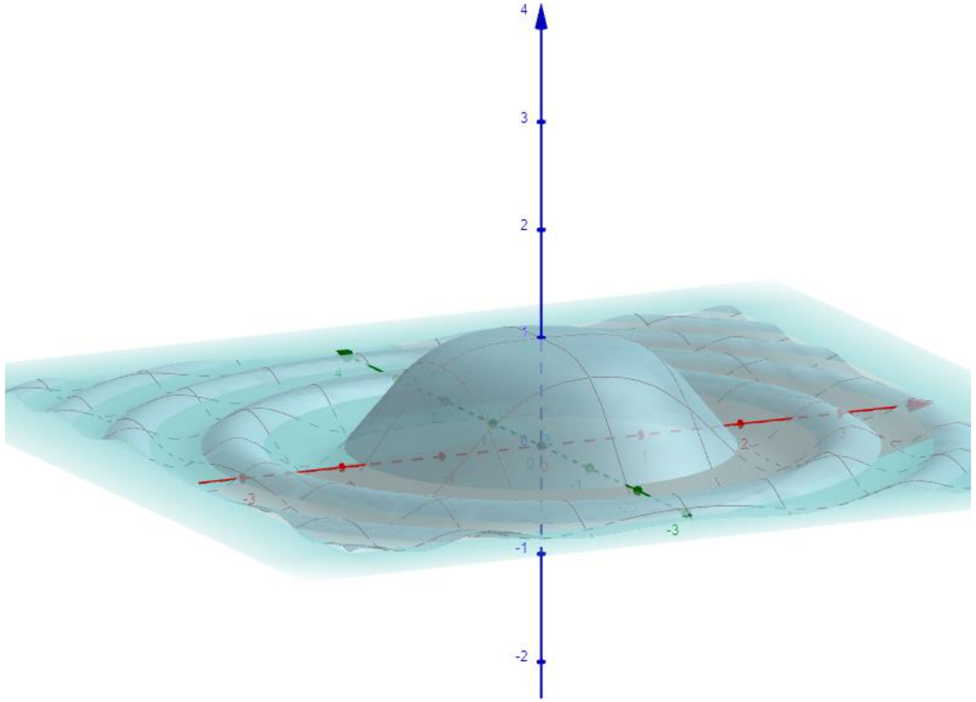
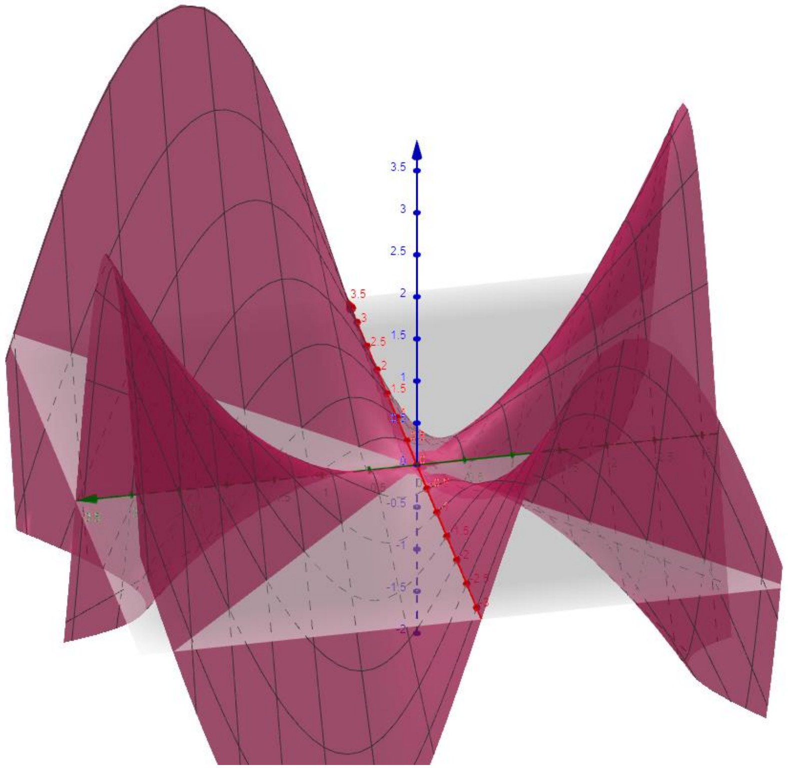


More 3D plotting examples

- https://www.geogebra.org/3d?lang=en
  - $z = 0$
  - $f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}$ f(x,y)=x*y(x^2-y^2)/(x^2+y^2)



Try it out yourself:

- https://www.geogebra.org/3d?lang=en
- $f(x,y) = x^2 + y^2$
  `f(x,y)=x^2+y^2`
- $f(x,y) = 5$
  `f(x,y)=5`
- $f(x,y) = x^2 - y^2$
  `f(x,y)=x^2-y^2`
- $f(x,y) = x^2 + y^2 + \frac{1}{x^2+y^2}$
  `f(x,y)=x^2+y^2+1/(x^2+y^2)`
- $f(x,y) = \sin x + \cos y$
  `f(x,y)=sin(x)+cos(y)`