Partial derivatives
Lecture 5b – 2021-06-09

MAT A35 – Summer 2021 – UTSC

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What is a derivative?

- A derivative measures the rate of change of a function as the variable it depends on changes.
- Given a function $f: \mathbb{R} \to \mathbb{R}$ written as $f(x)$, $\frac{df}{dx} = f'$ measures how quickly f changes when x changes.
- Note $f': \mathbb{R} \to \mathbb{R}$ since $f'(x)$ is a real number.

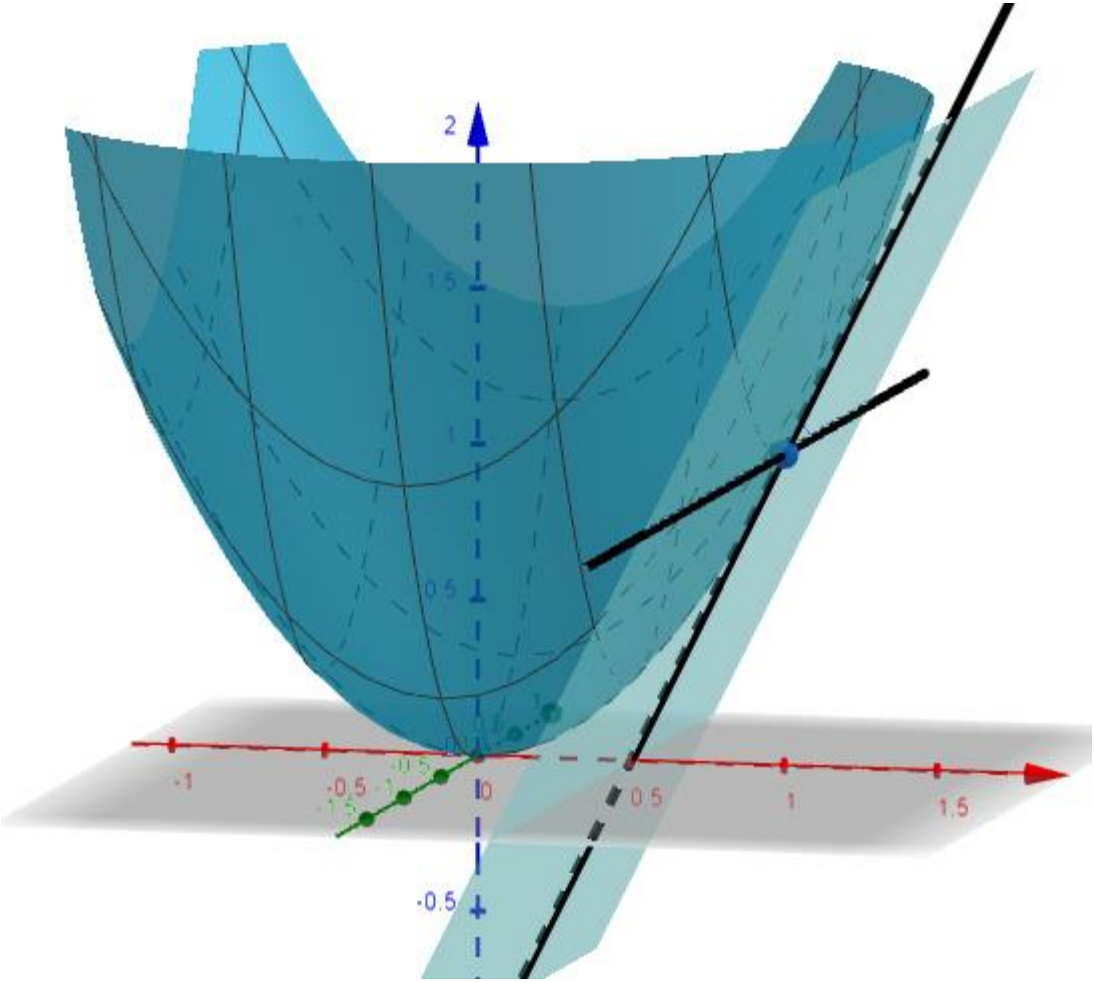
Partial derivatives of multivar. functions

- We can measure the rate of change of the function with respect to each variable independently, assuming the other variable doesn’t change.
- Given a function $f: \mathbb{R}^2 \to \mathbb{R}$ written as $f(x, y)$, the *partial derivative* $\frac{\partial f}{\partial x}$ measures how quickly f changes when x changes but y is fixed constant.
- Similarly, the partial derivative $\frac{\partial f}{\partial y}$ measures how quickly f changes when y changes but x is a fixed constant.
- Note $\frac{\partial f}{\partial x}: \mathbb{R}^2 \to \mathbb{R}$ takes as input a pair (x, y) and outputs a number

Pronunciation note: $\frac{\partial f}{\partial x}$ can be read several ways:

- del eff by del ecks
- del eff over del ecks
- del eff del ecks
- partial of eff with respect to ecks
- Sometimes even “dee eff dee ecks” if unambiguous

$$f(x, y) = x^2 + y^2$$



https://www.geogebra.org/3d/j8ntyjzw

Formal definition of partial derivatives

- Recall: for $z = f(x)$, where $f: \mathbb{R} \to \mathbb{R}$, a 1-variable function
  - $\frac{dz}{dx} = \frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
- Let $z = f(x, y)$, where $f: \mathbb{R}^2 \to \mathbb{R}$, a 2-variable function.
  - $\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$
  - $\frac{\partial z}{\partial y} = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$
- This generalizes in the natural way to n-variable functions, where you just treat all the other variables as constant.

Computing partial derivatives

- For the partial derivative with respect to a variable, treat all the other variables as constants and apply the normal derivative rules.

Example: $f(x, y) = x^2 + 2xy^2 + y^3$

Try it out

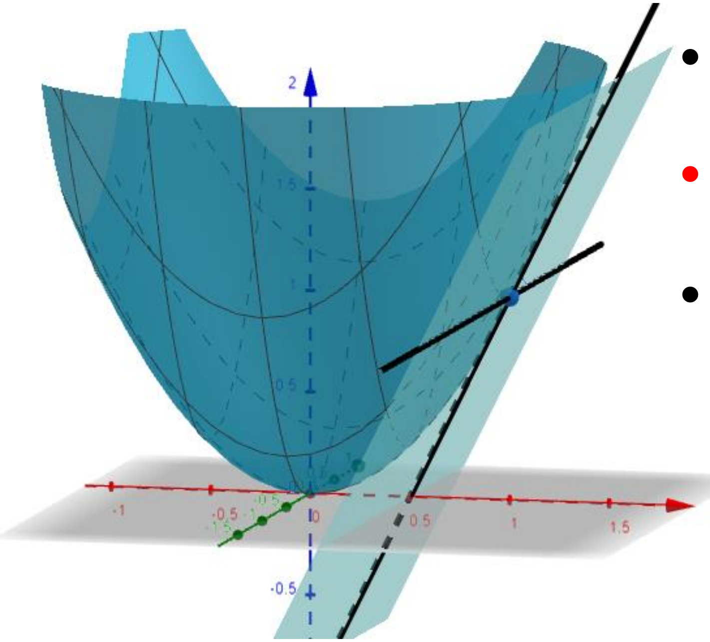
- $f(x, y) = 3x^2y + xy^2$
  - Compute $\frac{\partial f}{\partial x}$
  - Compute f_y
- $w = g(x, y, z) = 5y^2 + 2yz$
  - Compute $\frac{\partial g}{\partial x}(x, y)$
  - Compute g_y
  - Compute $\frac{\partial w}{\partial z}$
- Evaluating at a point
  - Compute $f_y(1,2)$
  - Compute $\frac{\partial w}{\partial z}(0,1,2)$

A: 0
B: $6xy + y^2$
C: $3x^2 + 2xy$
D: $3x^2 + 6xy + y^2 + 2xy$
E: None of the above

A: 0
B: $2y$
C: $10y + 2z$
D: $5y^2 + 2yz$
E: None of the above

A: 0
B: 2
C: 5
D: 7
E: None of the above

What about other directions?



- We had an entire tangent plane.
- $\frac{\partial f}{\partial x}$ says how fast f grows in the x-direction.
- $\frac{\partial f}{\partial y}$ says how fast f grows in the y-direction.
- <u>Advanced (not on quiz 3):</u>
- Given a direction vector $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ where $u_1^2 + u_2^2 = 1$, we can compute how quickly f grows in the u-direction by computing the matrix product

$$\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{\partial f}{\partial x} \cdot u_1 + \frac{\partial f}{\partial y} \cdot u_2$$

where $\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}$ is the gradient of f.

Jacobian matrix

- Consider a function $h: \mathbb{R}^2 \to \mathbb{R}^2$ that takes a point in the plane to another point in the plane.
- We can write $h\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$, where $f, g: \mathbb{R}^2 \to \mathbb{R}$.
- Then the Jacobian matrix of h (or of the pair of functions f and g) is given by:

$$J(x,y) = \begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$$

- The Jacobian matrix is the higher-dimensional analogue of a derivative, and tells you how the output of the function (a vector) changes as you go in a particular direction.

Example

- Gradient ≈ "total" derivative of $f: \mathbb{R}^2 \to \mathbb{R}$ because it combines together all the partial derivatives.
- Jacobian ≈ "total" derivative of $f: \mathbb{R}^2 \to \mathbb{R}^2$ because it combines together all the partial derivatives.

Higher-order partial derivatives

- Given $f(x, y)$ a function of two variables, $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x}(x, y)$ is also a function of two variables.
- Define:
  - $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} = f_{xx}$, which is taking the partial derivative by x twice
  - $\frac{\partial^2 f}{\partial y dx} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = f_{xy}$, which is taking partial-x, then partial-y
  - $\frac{\partial^2 f}{\partial x dy} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = f_{yx}$, which is taking partial-y, then partial-x
  - $\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\frac{\partial f}{\partial y} = f_{yy}$, which is taking the partial derivative by y twice

Example: $f(x, y) = x^3y^2 + y\sin x + xe^y$

- Note: "usually" it is true that $\frac{\partial^2 f}{\partial y dx} = f_{xy} = f_{yx} = \frac{\partial^2 f}{\partial x dy}$.

Try it out

- $f(x,y) = x^2y^2 + 4xy$
- $\dfrac{\partial f}{\partial x}$
- $\dfrac{\partial f}{\partial y}$
- $\dfrac{\partial^2 f}{\partial x^2}$
- $\dfrac{\partial^2 f}{\partial y dx}$
- $\dfrac{\partial^2 f}{\partial x dy}$
- $\dfrac{\partial^2 f}{\partial y^2}$

A: $2x^2$
B: $2x^2y + 4x$
C: $2y^2$
D: $2xy^2 + 4y$
E: $4xy + 4$

Hessian matrix

- Hessian matrix corresponds to second derivative

Hessian matrix ≈ 2nd total derivative

- Say we have $f: \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y)$.
- 1st total derivative $\approx \nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}$
- We can think of $\nabla f: \mathbb{R}^2 \to \mathbb{R}^2$ by transposing $\nabla f^T = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}$
- Then we can the total derivative of ∇f by using the Jacobian, and we'll call that new matrix the Hessian of f.
- $\text{Hessian}(f) = \text{Jacobian}(\nabla \mathrm{f}) = \text{Jacobian}\left(\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}\right)$

$$= \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y dx} \\ \frac{\partial^2 f}{\partial x dy} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

- The Hessian includes all the 2nd partial derivatives of f.