Multivariable integration
Lecture 5c – 2021-06-09

MAT A35 – Summer 2021 – UTSC

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Double definite integrals

limits of integration

$$\underbrace{\int_a^b}\, f(x)\,\underbrace{dx} = \int_{x=a}^{x=b} f(x)\,dx$$

variable of integration

$$\int_a^b \int_c^d f(x,y)\,dx\,dy = \int_{y=a}^{y=b} \int_{x=c}^{x=d} f(x,y)\,dx\,dy$$

integration over x

integration over y

Definite integration removes the variable

$$\int_1^2 x^2\,dx = \left[\frac{1}{3}x^3\right]\Bigg|_{x=1}^{x=2} = \frac{8}{3} - \frac{1}{3} = \frac{7}{8}$$

$$\left(\int x^2\,dx = \frac{1}{3}x^3 + C \qquad \text{indefinite} \right)$$

$$\int_1^y x^2\,dx = \left[\frac{1}{3}x^3\right]\Bigg|_{x=1}^{x=y} = \frac{y^3}{3} - \frac{1}{3}$$

Multiple definite integrals example

$$\int_{y=a}^{y=b} \int_{x=c}^{x=d} f(x, y)\, dx\, dy$$

- First integrate the inside integral, assuming all other variables are constant.
  - $\int_{x=c}^{x=d} f(x, y)\, dx = g(y)$ (because we got rid of the x-variable)
- Then integrate the outside variable, to get an answer
  - $\int_{y=a}^{y=b} g(y)\, dy =$ answer

Example: $\overbrace{\int_0^2 \underbrace{\int_{-1}^2 10xy^2\,dx}_{\text{treat } y \text{ as constant}}\,dy} = 40$

- $\int_{-1}^{2} 10xy^2\,dx = \left[5x^2y^2\right]\Big|_{x=-1}^{x=2} = 20y^2 - 5y^2 = 15y^2$

- $\int_{0}^{2} 15y^2\,dy = \left[5y^3\right]\Big|_{y=0}^{y=2} = 5\cdot 8 = 40$

Try it out

- $\int_0^2 \int_{-1}^1 2y\,dx\,dy$
  - $\int_{-1}^{1} 2y\,dx = [2xy]\Big|_{x=-1}^{x=1} = 2(1)y - 2(-1)y = 4y$
  - $\int_0^2 4y\,dy = [2y^2]\Big|_{y=0}^{y=2} = 8 - 0 = 8$

A: 2
B: 4
C: 8
D: 16
E: None

Switching order of integrals?

solve that

- Last slide: $\boxed{\int_0^2} \boxed{\int_{-1}^1 2y\,dx}\,dy = 8$
- What about: $\int_{-1}^1 \boxed{\int_0^2} 2y\,dy\,dx = \int_{x=-1}^{x=1} \int_{y=0}^{y=2} 2y\,dy\,dx$

$$\int_0^2 2y\,dy = [y^2]\Big|_{y=0}^{y=2} = 4 - 0 = 4$$

$$\int_{-1}^1 4\,dx = 4x\Big|_{-1}^{1} = 4 - (-4) = 8$$

- Often, we can switch the order of integration and get the same answer, but this is not always true.

Variables in the limits of integration

- We can use the outside integral variable in the limits of the inside variable (but not the other way around).

Ex. $$\int_0^1 \int_x^{x^2} xy^2\,dy\,dx = \int_0^1 \quad dx$$

when doing, treat x as constant

- $$\int_{y=x}^{y=x^2} xy^2\,dy = \left[\frac{1}{3}xy^3\right]\Bigg|_{y=x}^{y=x^2} = \frac{x^7}{3} - \frac{x^4}{3}$$
- $$\int_0^1 \left(\frac{x^7}{3} - \frac{x^4}{3}\right)dx = \frac{1}{3}\int_0^1 \left[x^7 - x^4\right]dx = \frac{1}{3}\left[\frac{1}{8}x^8 - \frac{1}{5}x^5\right]\Bigg|_{x=0}^{x=1}$$

$$= \frac{1}{3}\left[\frac{1}{8} - \frac{1}{5}\right] = \frac{1}{3}\left[\frac{5}{40} - \frac{8}{40}\right] = \frac{1}{3}\cdot\frac{-3}{40} = -\frac{1}{40}$$

Try it out

- $\int_0^2 \int_1^{y^2} 2x\,dx\,dy$

  - $\int_{x=1}^{x=y^2} 2x\,dx = \left[x^2\right]\Big|_1^{y^2} = y^4 - 1$

  - $\int_0^2 \quad dy = \int_0^2 (y^4 - 1)\,dy = \left[\frac{1}{5}y^5 - y\right]\Big|_0^2$

$$= \frac{32}{5} - 2 = \frac{22}{5}$$

A: $\frac{11}{5}$

B: $\frac{22}{5}$

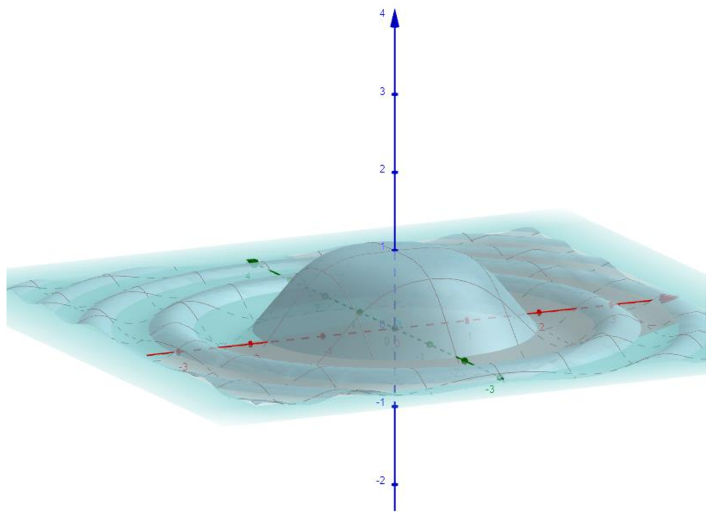
C: $\frac{33}{5}$

D: $\frac{44}{5}$

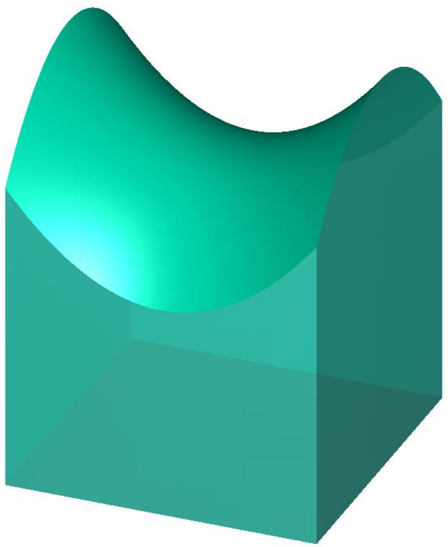
E: None

Geometric interpretation

- Multivariable functions $f: \mathbb{R}^2 \to \mathbb{R}$ can be thought of as surfaces.



- Double integrals correspond to the *volume* under the surface for a particular region



https://en.wikipedia.org/wiki/Multiple_integral