Critical points, maximums, and minimums
Lecture 5d – 2021-06-11

MAT A35 – Summer 2021 – UTSC

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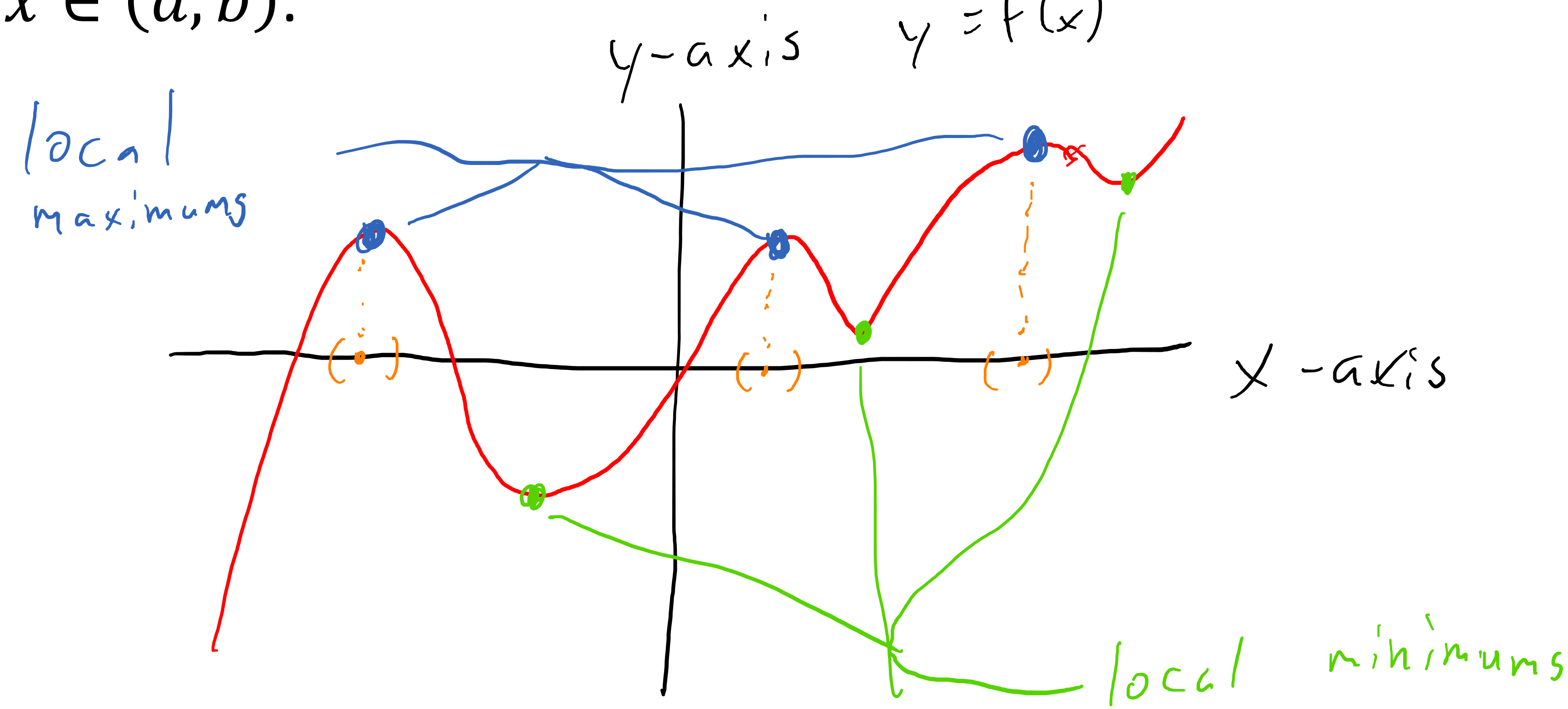
Local extrema of $f: \mathbb{R} \to \mathbb{R}$

- Given a function $f: \mathbb{R} \to \mathbb{R}$, x_0 is a local/relative minimum (or maximum) of f if there exists a small neighborhood (a, b) around x_0 where $f(x) > f(x_0)$ (resp. $f(x) < f(x_0)$) for any $x \in (a, b)$.



Critical points of $f: \mathbb{R} \to \mathbb{R}$

- Given a differentiable function $f: \mathbb{R} \to \mathbb{R}$, x_0 is a critical point of f if $f'(x_0) = 0$.

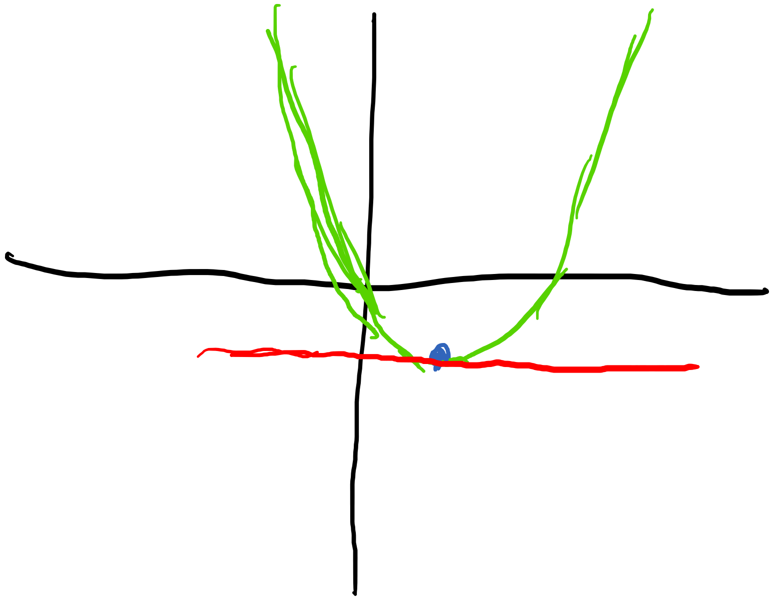
Ex. $f(x) = x^2 - 2x$

$f'(x) = 2x - 2$

$2x_0 - 2 = 0$

$\Rightarrow x_0 = 1$

$f'(1) = 0$



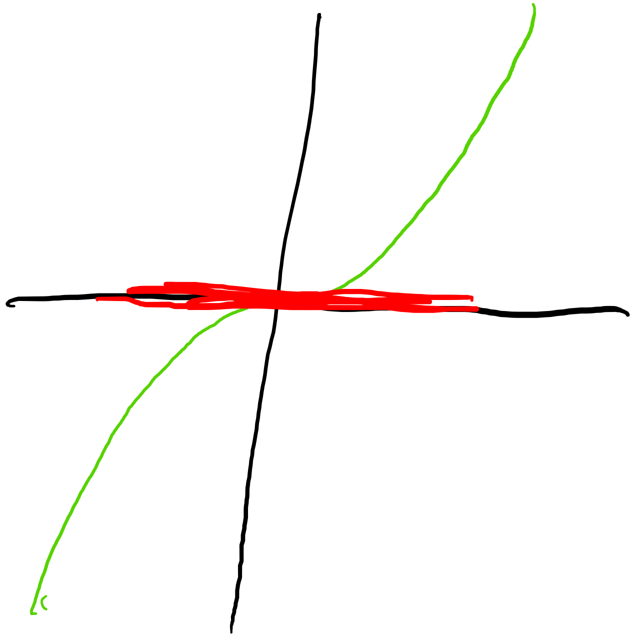
Ex. $f(x) = x^3$

$f'(x) = 3x^2$

$3x_0^2 = 0$

$x_0 = 0$

$f'(0) = 0$

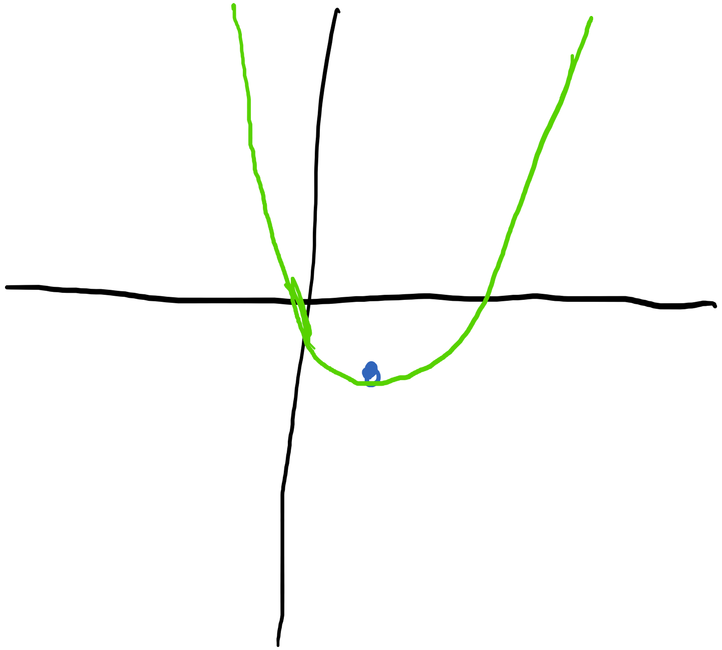


2^{nd} derivative test

- Consider a twice-differentiable function $f: \mathbb{R} \to \mathbb{R}$ given by $f(x)$.
  - For any $x \in \mathbb{R}$ where $f'(x) = 0$ and $f''(x) > 0$, x is a local minimum.
  - For any $x \in \mathbb{R}$ where $f'(x) = 0$ and $f''(x) < 0$, x is a local maximum.
  - We do not have enough information if $f'(x) = 0$ and $f''(x) = 0$.

Ex.

$f(x) = x^2 - 2x$



$f'(x) = 2x - 2 = 0$

crit pt.: $x = 1$

$f''(x) = 2$

$\Rightarrow f''(1) = 2$

$\Rightarrow$ 1 is a local min.

Try it out

- Find the only critical point of $f(x) = -(x-1)^2$. Determine if it is a local minimum, a local maximum, or neither?

$$f(x) = -(x-1)^2$$

$$f'(x) = -2(x-1)$$

Crit. pt: $-2(x_0 - 1) = 0$

$$\Rightarrow x_0 = 1$$

$$f''(x) = -2$$

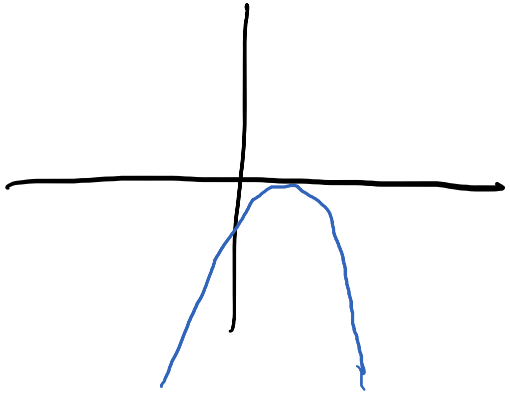
$$f''(1) = -2$$

Thus $x_0 = 1$ is a local maximum

A: Local minimum

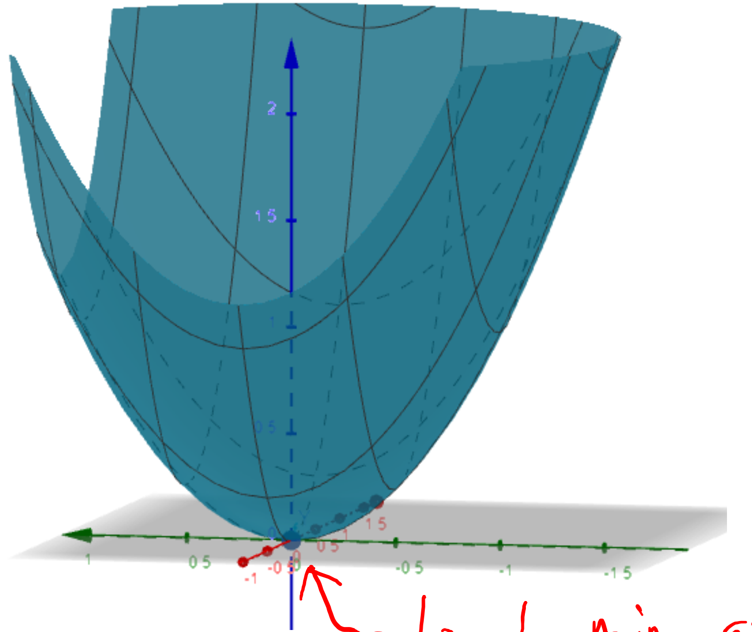
B: Local maximum

E: Neither



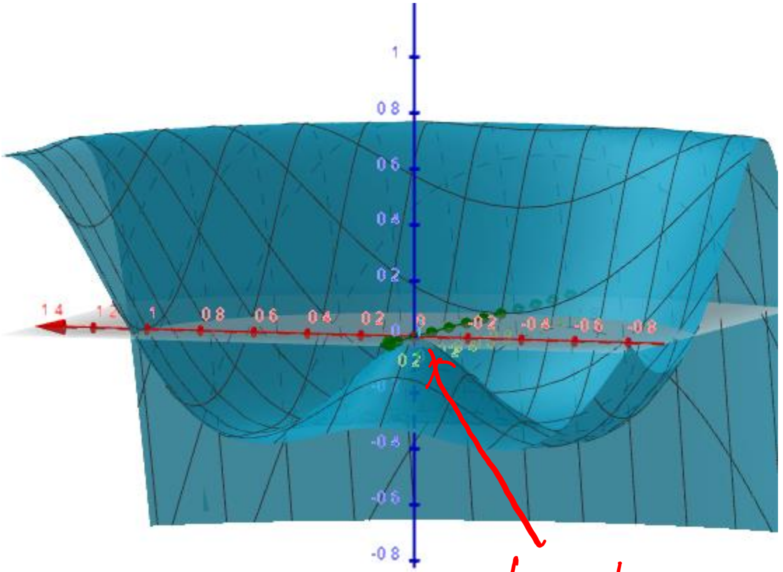
Local extrema of $f: \mathbb{R}^2 \to \mathbb{R}$

- Given a function $f: \mathbb{R}^2 \to \mathbb{R}$, (x_0, y_0) is a local/relative minimum (or maximum) of f if there exists a small rectangular neighborhood N around (x_0, y_0) where $f(x,y) > f(x_0, y_0)$ (resp. $f(x,y) < f(x_0, y_0)$) for any $(x,y) \in N$.



local min at $(0,0)$

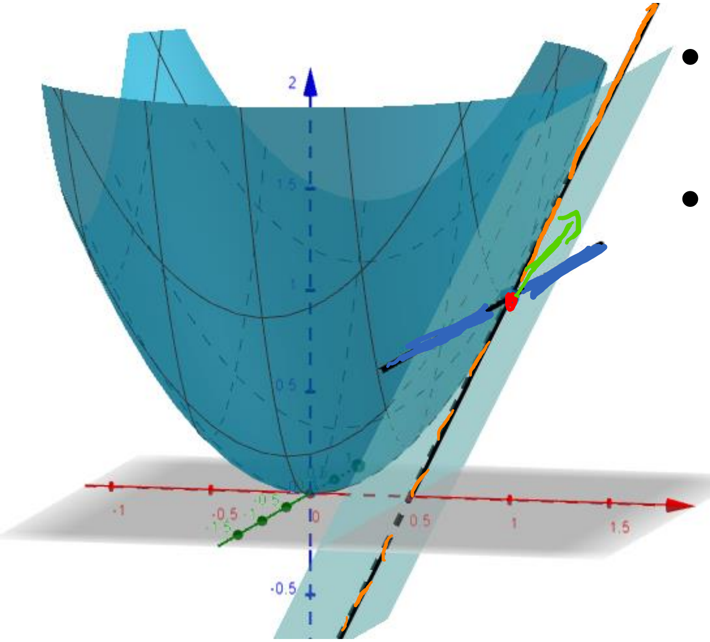
$f(x,y) = x^2 + y^2$



local max at $(0,0)$

$f(x,y) = \log_e(x^2 + y^2) \sin(x^2 + y^2)$, $f(0,0)=0$

Derivatives in higher dimensions



- We had an entire tangent plane.
- $f_x = \frac{\partial f}{\partial x}$ says how fast f grows in the x-direction.
- $f_y = \frac{\partial f}{\partial y}$ says how fast f grows in the y-direction.
- Given a direction vector $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ where $u_1^2 + u_2^2 = 1$, we can compute how quickly f grows in the u-direction by computing the matrix product

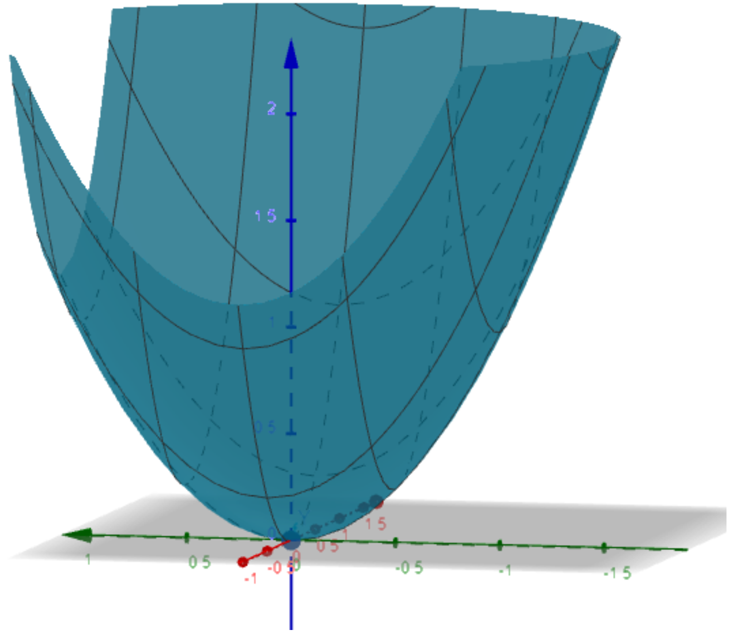
$$\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{\partial f}{\partial x} \cdot u_1 + \frac{\partial f}{\partial y} \cdot u_2$$

where $\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}$ is the gradient of f.

Critical points of $f: \mathbb{R}^2 \to \mathbb{R}$

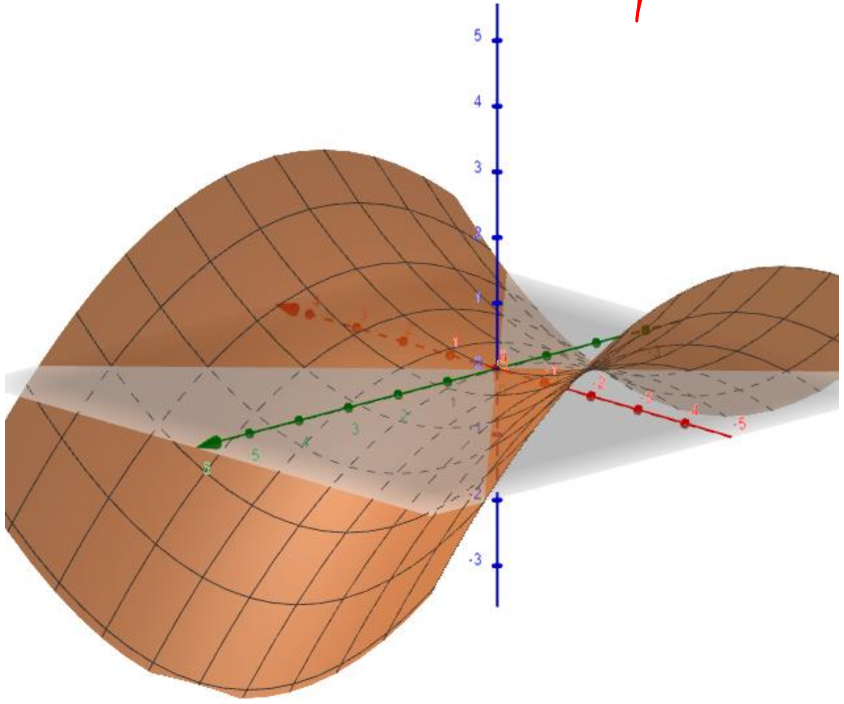
- Consider a differentiable function $f: \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y)$. (x_0, y_0) is a critical point of f if $f_x = 0$ and $f_y = 0$.

local min



$f(x, y) = x^2 + y^2$

saddle pt

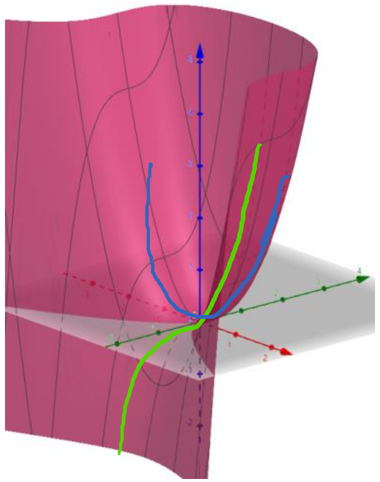


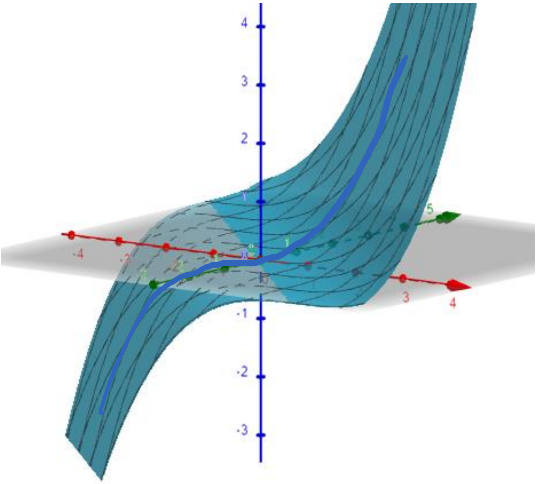
$$f(x, y) = \frac{x^2 - y^2}{10}$$

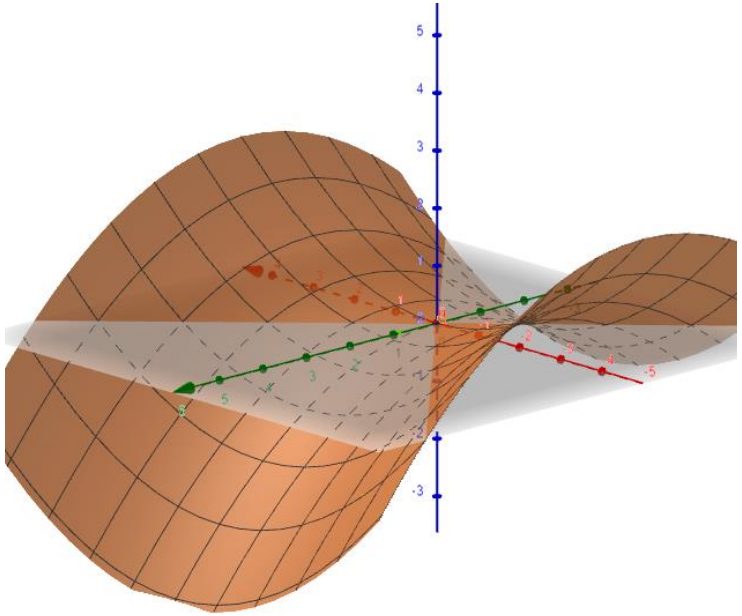
Saddle points

- Saddle points are critical points which are not local extrema.
- Prototypical example looks like a horse riding saddle because along one axis it goes down in both directions, and along the other axis, it goes up in both directions.
- Other examples may look quite different.

https://catscustomsaddlepads.com/wp-content/uploads/2017/10/TexInGlitterGold_HS2-1024x768.jpg







https://www.geogebra.org/3d/avydru8s

Higher-dimensional 2nd derivative test??

- For a single variable function, if x_0 is a critical point, we just need to check if $f''(x_0)$ is positive or negative to determine if minimum or maximum.

- Is there an analogous test for a critical point (x_0, y_0) of a multivariable function $f(x, y)$?

Attempt 1: all partial derivatives

- Consider a function $f: \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y)$. We have four second-order partial derivatives $f_{xx}, f_{xy}, f_{yx}, f_{yy}$.
- What if all four 2nd-order partial derivatives are positive?
- Try it out. The below functions have critical points at $(0,0)$. Use Geogebra to classify the critical point.
  - $f(x, y) = x^2 + xy + y^2$ $f_x = 2x + y$ $f_y = x + 2y$

    $f_{xx} = 2$ $f_{xy} = 1$ $f_{yx} = 1$ $f_{yy} = 2$
  - $f(x, y) = x^2 + 2xy + y^2$ $f_x = 2x + 2y$ $f_y = 2x + 2y$

    $f_{xx} = 2$ $f_{xy} = 2$ $f_{yx} = 2$ $f_{yy} = 2$
  - $f(x, y) = x^2 + 4xy + y^2$

    $f_{xx} = 2$ $f_{xy} = 4$ $f_{yx} = 4,$ $f_{yy} = 2$

A: Minimum
B: Maximum
C: Saddle Point

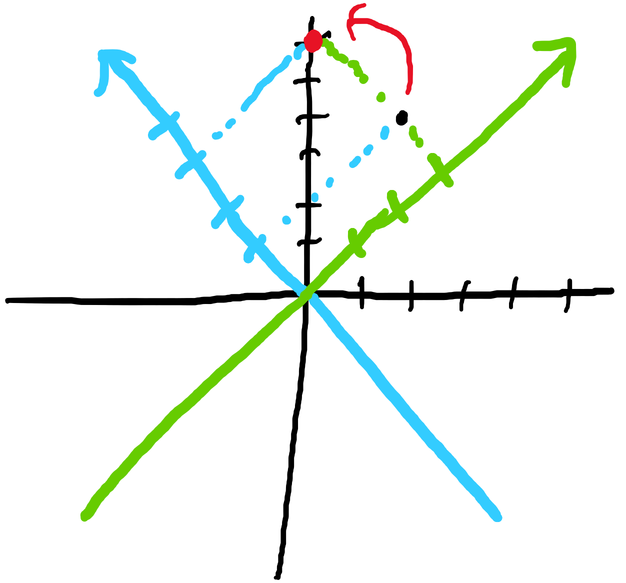
What about the Hessian matrix?

- Analogous to 2^{nd}-order total derivative.
- $H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y dx} \\ \frac{\partial^2 f}{\partial x dy} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$
- What does it mean for a matrix to be "positive"?
- What does it mean for a matrix to be "negative"?
- Answer depends on whether we are talking about matrix addition or matrix multiplication.

Eigenpairs show what happens to the axes

- If we have n distinct eigenpairs of an $n \times n$ matrix A, we can interpret the "action" of A by what it does to the eigenvectors.

$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ has eigenpairs $\left(3, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$, $\left(1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$



Note $\begin{bmatrix} 2 \\ 4 \end{bmatrix} = 1 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + 3 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$A \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 1 \cdot A \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + 3 \cdot A \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= 3 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + 3 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

Hessian test: eigenvalue signs

- Given a twice-differentiable function $f: \mathbb{R}^2 \to \mathbb{R}$, let the Hessian $H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$. If (x_0, y_0) is a critical point of f (i.e. $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$), then f is a local minimum if all eigenvalues of H are positive.
- f is a local maximum if all eigenvalues of H are negative.
- f is a saddle point if one eigenvalue is positive and one eigenvalue is negative.
- We do not have enough information if any eigenvalue is zero.

Example

- $f(x, y) = x^2 + xy + y^2$

$f_x = 2x + y = 0$

$f_y = x + 2y = 0$

$x = 0$

$y = 0$

Crit. pt. at $(0, 0)$

$f_{xx} = 2$

$f_{xy} = f_{yx} = 1$

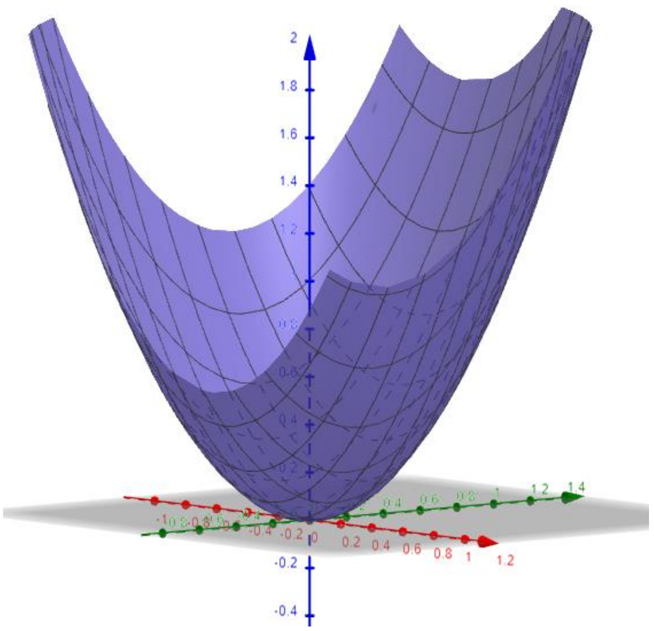
$f_{yy} = 2$

$H = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$f_{xx}(0,0) = 2$

$f_{xy}(0,0) = 1$

$\vdots$



$H(0,0) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 4 - 4\lambda + \lambda^2 - 1 = 0$

$\lambda^2 - 4\lambda + 3 = 0$

$(\lambda - 1)(\lambda - 3) = 0$

$\lambda = 1, 3$

two pos eigenvals, so local minimum

Example

- $f(x, y) = x^2 + 2xy + y^2$

crit pt at $(0,0)$

$f_{xx} = 2 \quad f_{xy} = f_{yx} = 2 \quad f_{yy} = 2$

$$H(0,0) = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

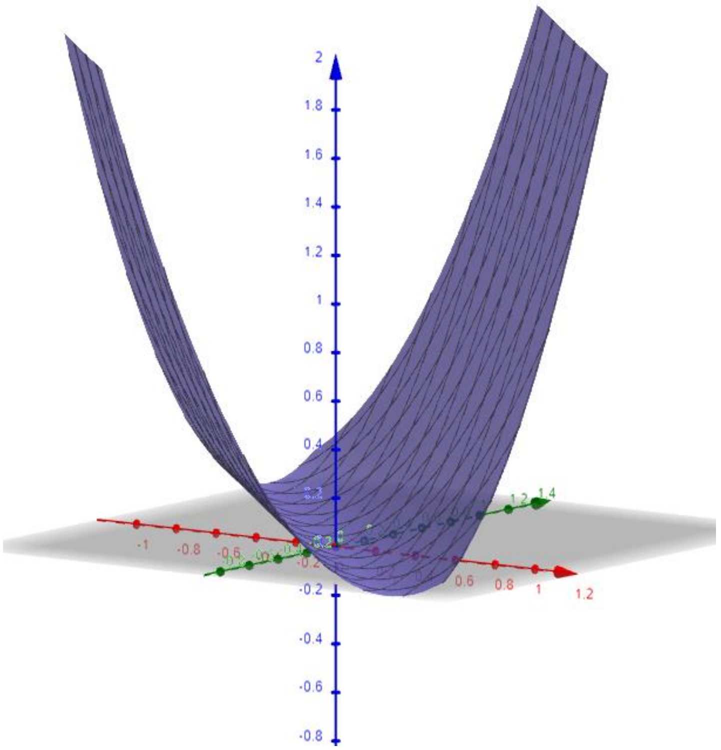
$$\begin{vmatrix} 2-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} = 4 - 4\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 - 4\lambda = 0$$

$$\lambda(\lambda - 4) = 0$$

$$\lambda = 0, 4$$

Cannot say using Hessian test because one eigenvalue is 0



Example

- $f(x, y) = x^2 + 4xy + y^2$

$f_x = 2x + 4y = 0$

$f_y = 4x + 2y = 0$

$\Rightarrow x = 0, \ y = 0$ is crit. pt.

$f_{xx} = 2, \quad f_{xy} = f_{yx} = 4 \quad f_{yy} = 2$

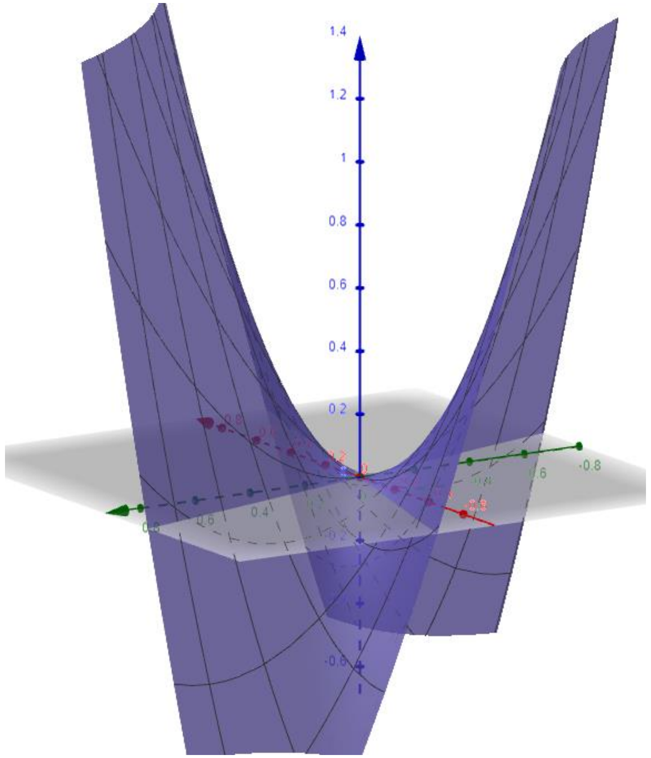
$H(0,0) = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$

$\begin{vmatrix} 2-\lambda & 4 \\ 4 & 2-\lambda \end{vmatrix} = 4 - 4\lambda + \lambda^2 - 16 = 0$

$\lambda^2 - 4\lambda - 12 = 0$

$(\lambda - 6)(\lambda + 2) = 0$

$\lambda = 6, -2$

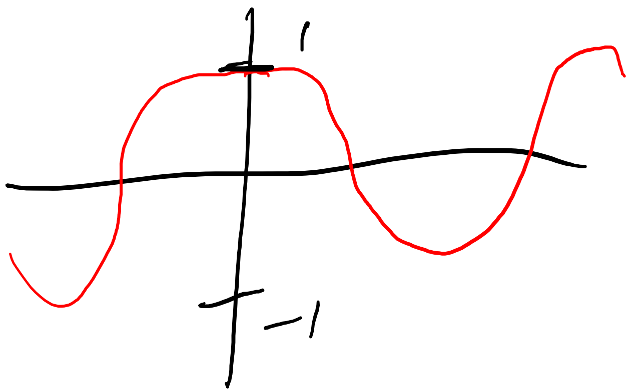


saddle pt because 1 direction pos and other direction neg

Example with nonconstant Hessian

- $f(x,y) = \sin(x^2+y^2) + 1.1(x^2+y^2)$

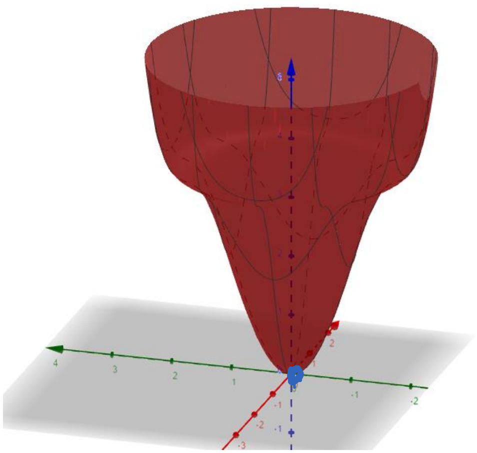
$$f_x = 2x\cos(x^2+y^2) + 1.1\cdot 2x$$
$$= 2x\left[1.1 + \cos(x^2+y^2)\right]$$
$$f_y = 2y\cos(x^2+y^2) + 1.1\cdot 2y$$
$$= 2y\left[\underbrace{1.1 + \cos(x^2+y^2)}_{\neq 0}\right]$$



Solve:
$$\begin{cases} 2x\left[1.1+\cos(x^2+y^2)\right] = 0 \\ 2y\left[1.1+\cos(x^2+y^2)\right] = 0 \end{cases}$$

$$\Rightarrow \begin{cases} 2x = 0 \\ 2y = 0 \end{cases} \Rightarrow \begin{matrix} x=0 \\ y=0 \end{matrix} \text{ at crit pt.}$$

Crit. pt. at $(x_0, y_0) = (0,0)$



Example continued (2nd order partials)

- $f(x, y) = \sin(x^2 + y^2) + 1.1(x^2 + y^2)$
- $f_x = 2x[1.1 + \cos(x^2 + y^2)\,]$
- $f_y = 2y[1.1 + \cos(x^2 + y^2)\,]$

$f_{xx} = 2x \cdot 2x \cdot (-\sin(x^2+y^2)) + 2[1.1 + \cos(x^2+y^2)]$

$= -4x^2 \sin(x^2+y^2) + 2[1.1 + \cos(x^2+y^2)]$

$f_{xx}(0,0) = 4.2$

$f_{xy} = f_{yx} = -4xy\sin(x^2+y^2)$ $\qquad f_{xy}(0,0) = f_{yx}(0,0) = 0$

$f_{yy} = -4y^2\sin(x^2+y^2) + 2[1.1 + \underbrace{\cos(x^2+y^2)}_{1}]$

$f_{yy}(0,0) = 4.2$

Example continued (Hessian)

- $f_{xx}(0,0) = \overset{4.2}{\cancel{2.2}}, f_{xy}(0,0) = f_{yx}(0,0) = 0, f_{yy}(0,0) = \overset{4.2}{\cancel{2.2}}$

$$H(0,0) = \begin{bmatrix} 4.2 & 0 \\ 0 & 4.2 \end{bmatrix}$$

Eigenvalues are 4.2 (two times), all positive

Thus (0,0) is a local minimum.

Try it out

- $f(x,y) = x^2 + xy + y^2 - 3x$
- What is the only critical point of f?

$f_x = 2x + y - 3 = 0$

$f_y = x + 2y = 0$

$\begin{cases} 2x + y = 3 \\ x + 2y = 0 \end{cases}$

$x = -2y$

$-4y + y = 3$

$-3y = 3$

$x = 2 \quad y = -1$

A: (-1, 2)
B: (-1, 1)
C: (1, -1)
D: (2, -1)
E: None

Try it out

- $f(x, y) = x^2 + xy + y^2 - 3x$
- Is the critical point a min, max, or saddle point?

Crit pt $(2, -1)$

$f_{xx} = 2$ $\quad f_{xx}(2,-1) = 2$ $\quad H(2,-1) =$

$f_{xy} = f_{yx} = 1$ $\quad f_{xy}(2,-1) = 1$ $\quad \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$f_{yy} = 2$ $\quad f_{yy}(2,-1) = 2$

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 4 - 4\lambda + \lambda^2 - 1 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda - 1)(\lambda - 3) = 0$$

$\lambda = 1, 3 \Rightarrow$ minimum

A: Minimum
B: Maximum
C: Saddle Point

Shortcut strategy: D-test

- Given a function $f(x, y)$, find the critical points by setting $f_x = 0$ and $f_y = 0$ simultaneously. Let (a, b) be a critical point.
- Let $D = f_{xx}(a, b) \cdot f_{yy}(a, b) - f_{xy}(a, b) \cdot f_{yx}(a, b)$.

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

  - Note that D is the determinant of the Hessian at the critical point.
  - f has a maximum at (a, b) if $D > 0$ and $f_{xx}(a, b) < 0$.
  - f has a minimum at (a, b) if $D > 0$ and $f_{xx}(a, b) > 0$.
  - f has a saddle point at (a, b) if $D < 0$.
  - We don't have enough information if $D = 0$.
- Advanced: This works because it turns out that the determinant is the product of the eigenvalues (counting repeats) and the sum of the diagonal elements (the "trace") equals the sum of the eigenvalues (counting repeats).

Example (from above)

$$H = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3$$

$$f_{xx} = 2$$

$\Big\}$ local minimum