Introduction to Ordinary Differential Equations
Lecture 7a – 2021-06-30

MAT A35 – Summer 2021 – UTSC

Prof. Yun William Yu

What is a differential equation?

- An equation relates variables (e.g. x, y, z), and a solution is a set of values that makes the equation true.

$x^2 + 2y + xz = 0$

Sol: $x = 1$, $y = -1$, $z = 1$

Sol: $x = 0$, $y = 0$, $z = 0$

- A differential equation relates variables and their derivatives, and a solution is a function that makes the equation true.

$5 = \frac{dy}{dx} = y'$

Sol: $y = 5x$

Sol: $y = 5x + 1$

$y = \frac{dy}{dx} = y'$

$y = y'$

Sol: $y = e^x$

$\frac{dy}{dx} = e^x$

Sol: $y = 100e^x$

$\frac{dy}{dx} = 100e^x$

Notational reminder

- The most unambiguous way to write derivatives is to write both variables:
  - $\frac{dy}{dx}$ is the derivative of y by the x variable.
  - $\frac{dx}{dt}$ is the derivative of x by the t variable.
- Primes/apostrophes denote the derivative by x
  - $y' = \cancel{\frac{df}{dx}}\ \frac{dy}{dx}$
  - $f'' = \frac{d^2f}{dx^2}$
  - $y^{(n)} = \frac{d^ny}{dx^n}$
- Dots above a variable denote a time-derivative by t
  - $\dot{x} = \frac{dx}{dt}$
  - $\ddot{y} = \frac{d^2y}{dt^2}$

ODEs vs PDEs

$y = f(x)$

$$y' = \frac{dy}{dx} \quad \begin{matrix} \leftarrow \text{dependent var.} \\ \leftarrow \text{independent var.} \end{matrix}$$

- Ordinary differential equations (ODEs) contain only one independent variable, so we have regular derivatives.

$$F(x, y, y', y'', y''', \dots, y^{(n)}) = 0 \quad \text{where} \quad y = f(x)$$

Ex. $x - y' = 0$, $x^2 y'' + xy' + xy^2 - 1 = 0$

- Partial differential equations (PDEs) contain multiple independent variables, so we have partial derivatives.

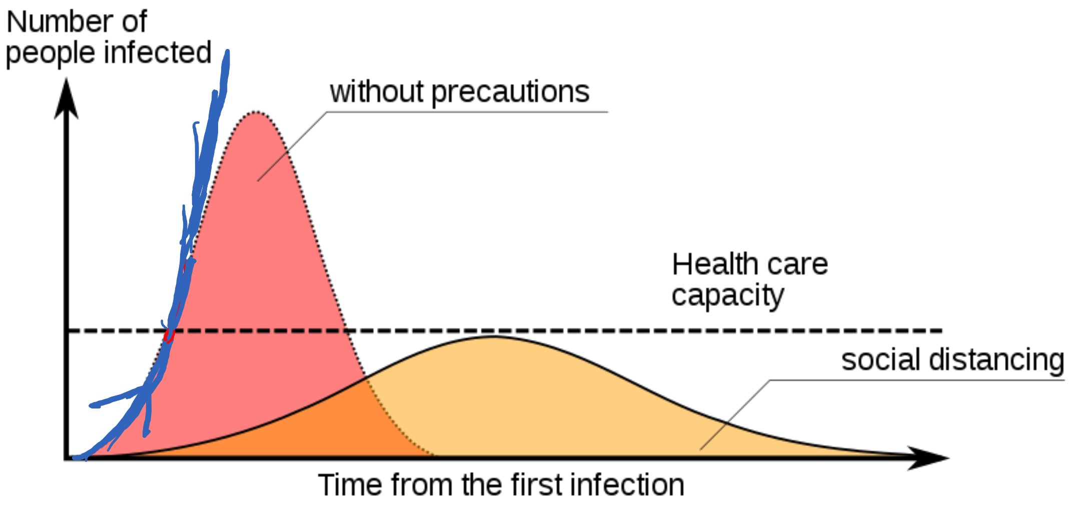
$$\text{dep.} \rightarrow z = f(\underbrace{x, y}_{\uparrow\ \uparrow\ \text{ind. var.}})$$

Ex. $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} - z^2 y = 0$

We will not be covering PDEs in MATA35 other than to recognize them.

Example of ODE

- Early on in an epidemic, the infection rate is proportional to the number of infected individuals.
- Let $I(t)$ be the number of infected individuals at time t



$$\dot{I}(t) = kI(t)$$

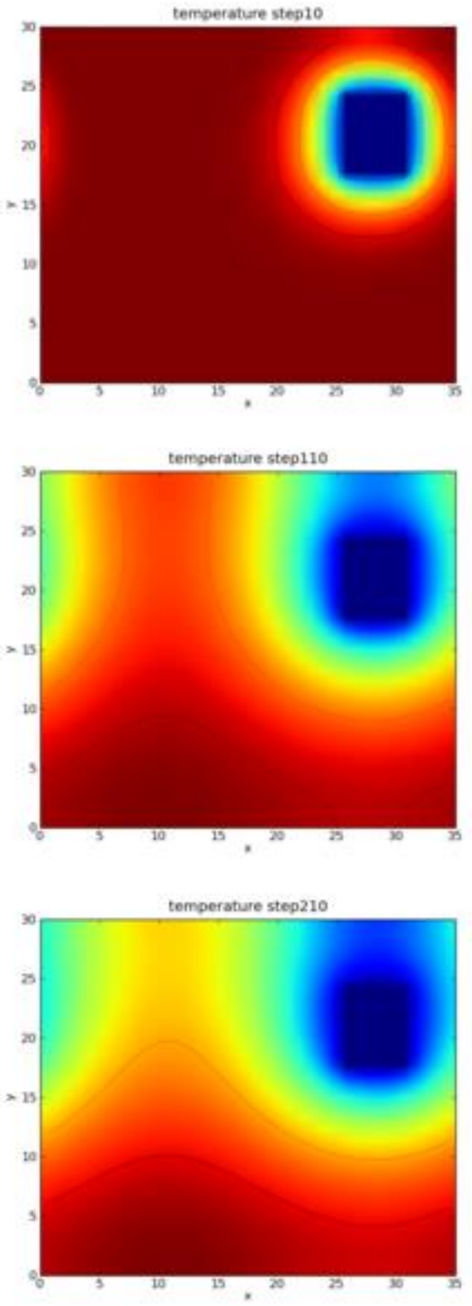
Sol: $I(t) = \underbrace{C}_{\text{initial infected number}} \underbrace{e^{kt}}_{\text{exponential growth}}$, where C is a constant

Example of PDE

- Heat diffusion has multiple spatial independent variables, and is normally modelled by a PDE.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$u(t, x, y, z)$ is a function of 4 ind. variables.



https://commons.wikimedia.org/wiki/File:Heat_diffusion.png

Classify the following

- $x^2 + 5x + y^2 = 5xy$
- $x^2 + 5y' + y^2 = 4xy'$
- $t^2\overset{\frac{dx}{dt}}{\dot{x}} + 5t = x$ ← ODE
- $\left(\frac{\partial z}{\partial x}\right)^2 + y^2\frac{\partial z}{\partial y} - 5x$
- $\left(\frac{\partial z}{\partial x}\right)^3 + \frac{\partial z}{\partial y} + z^2 = 2$

A: Typical equation
B: ODE
C: PDE
D: ???
E: None of the above

Time vs. space

- Often, in practice, ODEs describe how a system changes with time as the only independent variable (whether we call the independent variable “x” or “t”)
- Often, PDEs include how a system changes with space, and sometimes also with time, giving multiple independent variables.

Specific types of ODEs

nth order ODE

- General form of ODEs: $F\left(x, y, y', \dots, y^{(n)}\right) = 0$
- First-order ODE: ~~$F(x, y') = 0$~~ $F(x, y, y')=0$
- Pure-time ODE: $y' = f(x)$ ← only depends on x
- Autonomous ODE: $F\left(y, y', \dots, y^{(n)}\right) = 0$ ← no explicit x-dep.

  Ex. $y'=y$
- Autonomous 1st-order: $y' = f(y)$
- Linear ODEs: $a_n(x)y^{(n)} + \cdots + a_1(x)y' + a_0(x)y = q(x)$, where $a_i(x)$ and $q(x)$ are all functions of x.

  $5y' + xy = x$
- Linear 1st-order: $y' + p(x)y = q(x)$

If $a_1(x)y' + a_0(x)y = q_0(x)$

$$y' + \frac{a_0(x)}{a_1(x)}y = \frac{q_0(x)}{a_1(x)}$$

$$p(x) = \frac{a_0(x)}{a_1(x)}$$

$$q(x) = \frac{q_0(x)}{a_1(x)}$$

Try it out: classify order of ODE

- $y'' + y' + y^2 = 5$ 2nd order
- $y' - 1 = x^2$ 1st order
- $\frac{dx}{dy}\left(\frac{d^2x}{dy^2} + 1\right) = y$ 2nd order
- $y'''' + 4y'' + 4y = x^2$ 4th order
- $\dot{x} = t^2 + 1$ 1st order

$\dot{x}(\ddot{x}+1) = t$

$\dot{x}\ \ddot{x} \neq \dddot{x}$

$\left[\frac{d}{dx}\right]\left[\frac{d}{dx}\right]f$

$\left[\frac{d^2}{dx^2}\right]f$

$\frac{dy}{dx}$ dy ← dep. var. dx ← ind. vari

$\frac{dx}{dy}$ dx ← dep. var. dy ← ind. vari

$\frac{dx}{dt}$ dx ← dep. var. dt ← ind. var.

A: 1st order
B: 2nd order
C: 3rd order
D: 4th order
E: None of the above

Try it out: pure time/autonomous/neither

- $y'' + y' + y^2 = 5$

$y' = \frac{dy}{dx}$ Autonomous, no direct x-dependence

- $y' - 1 = x^2$

$y' = x^2 + 1$ Pure-time, y' depends only on x

- $\frac{dx}{dy}\left(\frac{d^2x}{dy^2} + 1\right) = \overset{\text{ind. var.}}{y}$

↑ ↑ two diff. derivatives, so not pure-time.

None of the above, because it depends on y and has two different derivatives

- $y'''' + 4y'' + 4y = x^2$

$y' = \frac{dy}{dx}$ ← dep. var. ← ind. var.

None, because depends on y'', y''', y, x

Ex. $y' = 0$

↑ both

- $\dot{x} = t^2 + 1$

$\dot{x} = \frac{dx}{dt}$ Pure-time.

$\dot{x}$ depends only on t.

A: Autonomous
B: Pure-time
C: Both of the above
D: ???
E: None of the above

Try it out: linear vs nonlinear

- $y'' + y' + y^2 = 5$

  nonlinear

  square of y $y^2 = y \cdot y$

- $y' - 1 = x^2$

  linear, because ind var x doesn't count

- $\frac{dx}{dy}\left(\frac{d^2x}{dy^2} + 1\right) = y$

  nonlinear, multiplying together two derivatives

- $\underline{y''''} + 4\underline{y''} + 4\underline{y} = x^2$

  linear, x^2 doesn't matter

- $\dot{x} = t^2 + 1$

  linear, t is indep var.

  t^2 doesn't matter

A: Linear

B: Nonlinear

[redacted]

D: ???

E: None of the above

$\sin y' + y = 0$

nonlinear