# Linear 1<sup>st</sup> order ODEs and Integrating Factors Lecture 7c – 2021-06-30

MAT A35 – Summer 2021 – UTSC

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#### **Existence-Uniqueness Theorem**

• Consider the 1<sup>st</sup> order linear ODE initial value problem

$$y' + p(x)y = q(x),$$
  $y(x_0) = y_0$ 

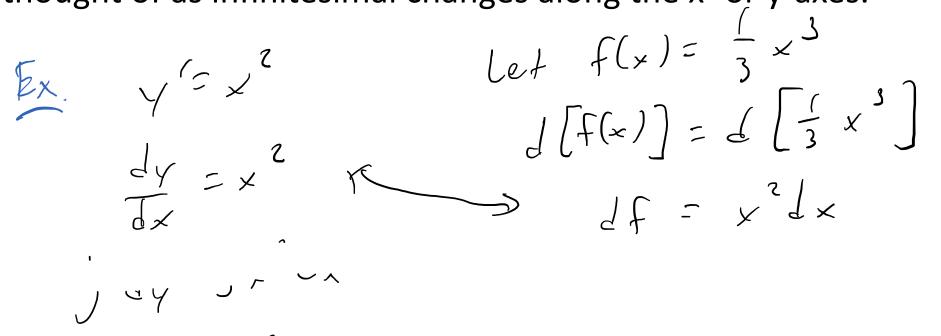
If p and q are continuous functions on an interval I containing x<sub>0</sub>, then there exists a unique solution to the IVP for every point in I.

 In a more theoretical ordinary differential equations class, a lot of time is spent on proving various existence theorems, uniqueness theorems, and existence-uniqueness theorems.

### Differentials

 $y = \frac{f}{2} x^3 + C$ 

• Differentials dx and dy are the intuition behind  $\frac{dy}{dx}$ , and can be thought of as infinitesimal changes along the x- or y-axes.



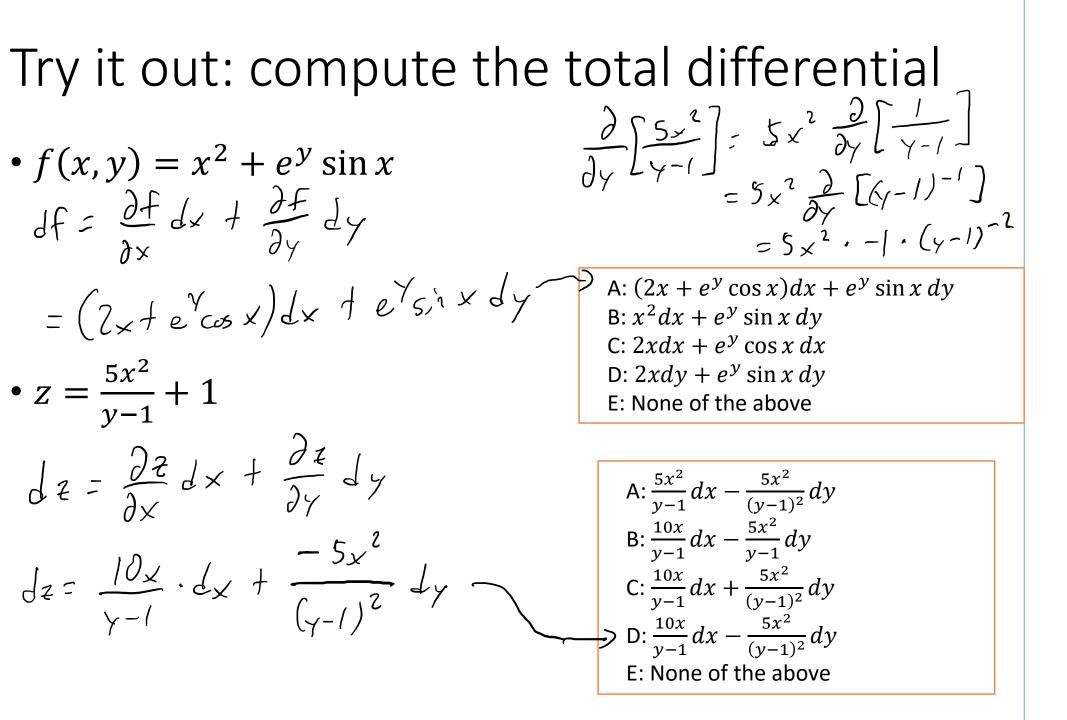
## Differentials of multi-variable functions

- Let z = f(x, y) be a function of both x and y. Recall that the gradient  $\nabla f = \begin{bmatrix} \partial f & \partial f \end{bmatrix}$ . • Recall that the gradient  $\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}$  gives the partial derivative in the  $u = \begin{bmatrix} u_x \\ u_y \end{bmatrix}$  direction by  $\nabla f \cdot u$ .
- We define the total differential of z by

$$dz = f_x(x, y)dx + f_y(x, y)dy = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}\frac{dy}{dy}$$

• i.e. 
$$dz = \nabla f \cdot \begin{bmatrix} ax \\ dy \end{bmatrix}$$
, where  $\nabla f$  is the **second ar**.

Example of total differential Z=x 2+2xy +44 Ex.  $dz = (2x + 2y) dx + (2x + 4y^{3}) dy$ dz = Zxdx + Zydx + Zxdy + 4y<sup>3</sup>dy  $f(x,y) = sin 2x + y^2 e^{x}$ Ex, df = 2 cos 2x dx + y<sup>2</sup>e<sup>x</sup>dx + 2ye<sup>x</sup>dy  $z = x^2 y^2$ EX.  $d_{z} = \frac{\partial}{\partial x} \left[ x^{2} y^{2} \right] d_{x} + \frac{\partial}{\partial y} \left[ x^{2} y^{2} \right] d_{y} \mathbf{m}$  $= 2 \times y^2 d \times + 2 \times^2 y d y$ 



#### Exact differential

• A differential

$$dz = P(x, y)dx + Q(x, y)dy$$

is an exact differential if there exists a function f(x, y) such that  $P(x, y) = \frac{\partial f}{\partial x}$  and  $Q(x, y) = \frac{\partial f}{\partial y}$ . Then z = f(x, y).

- Recall that for most nice functions  $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$ .
- Therefore, quick way to see if a differential is exact is to check if  $\frac{\partial P}{\partial Q} = \frac{\partial Q}{\partial Q}$

 $\begin{array}{c} \mu \mathcal{L} \mathcal{L} \\ d \mathcal{Z} = \frac{|0_{\chi}}{|\gamma|^{-1}} d\chi - \frac{S_{\chi}^{2}}{(\gamma - 1)^{2}} d\gamma \end{array}$ Example Ex. P(x,y) Q(x,y)Exact  $\frac{\partial Q}{\partial y} = \frac{-(\partial x)}{(y-1)^2}$  $\frac{\partial P}{\partial Y} = \frac{-10^{\times}}{(Y-1)^2}$  $dz = (5x^2+1)dx + xy^2 dy$ Ex. Q(x, y)P(x,y)  $\frac{\partial Q}{\partial x} = \sqrt{2}$  Not exact  $\frac{\partial P}{\partial y} = D$ 

Solving exact differential equations  $\left(\int \left( \frac{2}{x^{2}+1} \right) dy = x^{2} y^{2} + y^{2} + C(x) \right)$  $\frac{dy}{dz} = \frac{-lxy}{x^2 \pm l}$  $\int 2xy dx = x^2 y + C(y)$ (x+1)dy = -2xy dx Guess: f(x,y) = x 2 y + y e  $\int f = [2 \times y d \times f(x^2 + 1)dy] = D$ df= Zxy dx + (x2+1) dy) Q(x,y)P(7,4)  $\frac{\partial Q}{\partial r} = 2x = 7 exact | \int df = f + C$  $\frac{\partial P}{\partial Y} = 2 \times$  $\int dF = \int 0 = \int x^2 y + y + C = 0 \left[ \frac{dy}{dx} = \frac{-2x C}{(x^2 + 1)^2} \right]$ -124  $\frac{1}{x^{2}}$  $=) \gamma = \frac{C}{\sqrt{2} + C}$ 

### Integrating factors

• Sometimes, we can find an "integrating factor" I(x) to multiply by both sides of an inexact ODE to make it an exact ODE.

### Integrating factors

- Fact 1: every first-order ODE can be turned into an exact differential using an integrating factor.
- Fact 2: there is NO systematic way of guessing integrating factors for general ODEs.
- In MATA35, we will not expect you to use integrating factors outside of a few special cases where the integrating factors are known.

#### Integrating Factor for linear 1<sup>st</sup>-order ODE

• If you rewrite a linear 1<sup>st</sup> –order ODE in the following form:

$$y' + p(x)y = q(x)$$

which is equivalent to

$$dy + dx[p(x)y] = q(x)$$

• Then the integrating factor is  $e^{\int p(x)dx}$ 

#### General solution for 1<sup>st</sup>-order ODE

- If you rewrite a linear 1<sup>st</sup> –order ODE in the following form: y' + p(x)y = q(x)
- The general solution can be found by:
  - Determining the integrating factor  $I(x) = e^{\int p(x)dx} \swarrow \mathcal{I}(x)$
  - Multiply both sides by  $I(x): y' \cdot I(x) + p(x)y \cdot I(x) = q(x) \cdot I(x)$
  - Multiply both sides by dx:  $dy \cdot I(x) + p(x)y \cdot I(x)dx = q(x)I(x)dx$
  - The left hand side is the total differential d[I(x)y]
  - So we can integrate both sides to get  $I(x)y = \int q(x)I(x)dx$

• Then 
$$y = \frac{1}{I(x)} \left[ \int q(x) I(x) dx + C \right]$$

• In a single, ugly, long equation:  $y(x) = e^{-\int p(x)dx} \left[ \int e^{\int p(x)dx} dx + C \right]$  Try it out

• 
$$y' - 3x^2y = x^2$$

A: 
$$-\frac{1}{3} + e^{x^3} + C$$
  
B:  $-\frac{1}{3}e^{x^3} + C$   
C:  $-\frac{1}{3}e^{x^3 + C}$   
D:  $-\frac{1}{3} + Ce^{x^3}$   
E: None of the above