Linear 1st order ODEs and Integrating Factors Lecture 7c – 2021-06-30

MAT A35 – Summer 2021 – UTSC

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Existence-Uniqueness Theorem

- Consider the 1st order linear ODE initial value problem
$$y' + p(x)y = q(x), \qquad\qquad y(x_0) = y_0$$
- If p and q are continuous functions on an interval I containing x_0, then there exists a unique solution to the IVP for every point in I.

- In a more theoretical ordinary differential equations class, a lot of time is spent on proving various existence theorems, uniqueness theorems, and existence-uniqueness theorems.

Differentials

- Differentials dx and dy are the intuition behind $\frac{dy}{dx}$, and can be thought of as infinitesimal changes along the x- or y-axes.

Ex.

$y' = x^2$

$\frac{dy}{dx} = x^2$

$\int dy = \int x^2 dx$

$y = \frac{1}{3}x^3 + C$

Let $f(x) = \frac{1}{3}x^3$

$d[f(x)] = d\left[\frac{1}{3}x^3\right]$

$df = x^2 dx$

Differentials of multi-variable functions

$\partial f \leftarrow$ tiny change in f

$\partial x \leftarrow$ tiny change in x

- Let $z = f(x, y)$ be a function of both x and y.
- Recall that the gradient $\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}$ gives the partial derivative in the $u = \begin{bmatrix} u_x \\ u_y \end{bmatrix}$ direction by $\nabla f \cdot u$.
- We define the total differential of z by

$$dz = f_x(x, y)dx + f_y(x, y)dy = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

- i.e. $dz = \nabla f \cdot \begin{bmatrix} dx \\ dy \end{bmatrix}$, where ∇f is the ~~Jacobian~~. Gradient

Example of total differential

<u>Ex.</u> $z = x^2 + 2xy + y^4$

$$dz = (2x + 2y)\,dx + (2x + 4y^3)\,dy$$

$$dz = 2x\,dx + 2y\,dx + 2x\,dy + 4y^3\,dy$$

<u>Ex.</u> $f(x,y) = \sin 2x + y^2 e^x$

$$df = 2\cos 2x\,dx + y^2 e^x\,dx + 2y e^x\,dy$$

<u>Ex.</u> $z = x^2 y^2$

$$dz = \frac{\partial}{\partial x}\left[x^2 y^2\right] dx + \frac{\partial}{\partial y}\left[x^2 y^2\right] dy$$

$$= 2xy^2\,dx + 2x^2 y\,dy$$

Try it out: compute the total differential

- $f(x,y) = x^2 + e^y \sin x$

$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$

$= (2x + e^y \cos x) dx + e^y \sin x \, dy$

A: $(2x + e^y \cos x)dx + e^y \sin x \, dy$
B: $x^2 dx + e^y \sin x \, dy$
C: $2x dx + e^y \cos x \, dx$
D: $2x dy + e^y \sin x \, dy$
E: None of the above

$\frac{\partial}{\partial y}\left[\frac{5x^2}{y-1}\right] = 5x^2 \frac{\partial}{\partial y}\left[\frac{1}{y-1}\right]$

$= 5x^2 \frac{\partial}{\partial y}\left[(y-1)^{-1}\right]$

$= 5x^2 \cdot -1 \cdot (y-1)^{-2}$

- $z = \frac{5x^2}{y-1} + 1$

$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$

$dz = \frac{10x}{y-1} \cdot dx + \frac{-5x^2}{(y-1)^2} dy$

A: $\frac{5x^2}{y-1}dx - \frac{5x^2}{(y-1)^2}dy$
B: $\frac{10x}{y-1}dx - \frac{5x^2}{y-1}dy$
C: $\frac{10x}{y-1}dx + \frac{5x^2}{(y-1)^2}dy$
D: $\frac{10x}{y-1}dx - \frac{5x^2}{(y-1)^2}dy$
E: None of the above

Exact differential

- A differential

$$dz = P(x,y)dx + Q(x,y)dy$$

is an exact differential if there exists a function $f(x,y)$ such that $P(x,y) = \frac{\partial f}{\partial x}$ and $Q(x,y) = \frac{\partial f}{\partial y}$. Then $z = f(x,y)$.

- Recall that for most nice functions $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$.
- Therefore, quick way to see if a differential is exact is to check if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$$\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] \qquad \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right]$$

Example

Ex.

$$dz = \underbrace{\frac{10x}{y-1}}_{P(x,y)} dx - \underbrace{\frac{5x^2}{(y-1)^2}}_{Q(x,y)} dy$$

$$\frac{\partial P}{\partial y} = \frac{-10x}{(y-1)^2} \qquad \frac{\partial Q}{\partial y} = \frac{-10x}{(y-1)^2} \qquad \text{Exact}$$

Ex.

$$dz = \underbrace{(5x^2+1)}_{P(x,y)} dx + \underbrace{xy^2}_{Q(x,y)} dy$$

$$\frac{\partial P}{\partial y} = 0 \qquad \frac{\partial Q}{\partial x} = y^2 \qquad \text{Not exact}$$

Solving exact differential equations

$$\frac{dy}{dx} = \frac{-2xy}{x^2+1}$$

$$(x^2+1)dy = -2xy\,dx$$

$$df = \boxed{2xy\,dx + (x^2+1)dy} = 0$$

$$\underbrace{2xy}_{P(x,y)}\qquad \underbrace{(x^2+1)}_{Q(x,y)}$$

$$\frac{\partial P}{\partial y} = 2x \qquad \frac{\partial Q}{\partial x} = 2x \Rightarrow \text{exact}$$

$$\int (x^2+1)\,dy = x^2y + y + C(x)$$

$$\int 2xy\,dx = x^2y + C(y)$$

Guess: $f(x,y) = x^2y + y$

$$df = \boxed{2xy\,dx + (x^2+1)\,dy}$$

$$\int df = f + C$$

$$\int df = \int 0 \Rightarrow x^2y + y + C = 0$$

$$\Rightarrow y = \frac{C}{x^2+1}$$

$$\frac{dy}{dx} = \frac{-2xC}{(x^2+1)^2} = \frac{-2xy}{x^2+1}$$

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Integrating factors

- Sometimes, we can find an "integrating factor" $I(x)$ to multiply by both sides of an inexact ODE to make it an exact ODE.

Integrating factors

- Fact 1: every first-order ODE can be turned into an exact differential using an integrating factor.
- Fact 2: there is NO systematic way of guessing integrating factors for general ODEs.

- In MATA35, we will not expect you to use integrating factors outside of a few special cases where the integrating factors are known.

Integrating Factor for linear 1^{st}-order ODE

- If you rewrite a linear 1^{st} –order ODE in the following form:

$$y' + p(x)y = q(x)$$

  which is equivalent to

$$dy + dx[p(x)y] = q(x)$$

- Then the integrating factor is

$$e^{\int p(x)dx}$$

General solution for 1st-order ODE

- If you rewrite a linear 1st –order ODE in the following form:

$$y' + p(x)y = q(x)$$

- The general solution can be found by:
  - Determining the integrating factor $I(x) = e^{\int p(x)dx}$ $I(x)$
  - Multiply both sides by $I(x)$: $y' \cdot I(x) + p(x)y \cdot I(x) = q(x) \cdot I(x)$
  - Multiply both sides by dx: $dy \cdot I(x) + p(x)y \cdot I(x)dx = q(x)I(x)dx$
  - The left hand side is the total differential $d[I(x)y]$
  - So we can integrate both sides to get $I(x)y = \int q(x)I(x)dx$
  - Then $y = \frac{1}{I(x)}\left[\int q(x)I(x)dx + C\right]$
- In a single, ugly, long equation:

$$y(x) = e^{-\int p(x)dx}\left[\int e^{\int p(x)dx}dx + C\right]$$

Try it out

- $y' - 3x^2y = x^2$

A: $-\frac{1}{3} + e^{x^3} + C$
B: $-\frac{1}{3}e^{x^3} + C$
C: $-\frac{1}{3}e^{x^3+C}$
D: $-\frac{1}{3} + Ce^{x^3}$
E: None of the above