Linear 1st order ODEs and Integrating Factors Lecture 7c – 2021-06-30

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Existence-Uniqueness Theorem

- Consider the 1st order linear ODE initial value problem
$$y' + p(x)y = q(x), \qquad\qquad y(x_0) = y_0$$
- If p and q are continuous functions on an interval I containing x_0, then there exists a unique solution to the IVP for every point in I.

- In a more theoretical ordinary differential equations class, a lot of time is spent on proving various existence theorems, uniqueness theorems, and existence-uniqueness theorems.

Differentials

- Differentials dx and dy are the intuition behind $\frac{dy}{dx}$, and can be thought of as infinitesimal changes along the x- or y-axes.

Differentials of multi-variable functions

- Let $\mathrm{z} = f(x, y)$ be a function of both x and y.
- Recall that the gradient $\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}$ gives the partial derivative in the $u = \begin{bmatrix} u_x \\ u_y \end{bmatrix}$ direction by $\nabla f \cdot u$.
- We define the total differential of z by

$$dz = f_x(x, y)dx + f_y(x, y)dy = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

- i.e. $dz = \nabla f \cdot \begin{bmatrix} dx \\ dy \end{bmatrix}$, where ∇f is the Jacobian.

Example of total differential

Try it out: compute the total differential

- $f(x, y) = x^2 + e^y \sin x$

A: $(2x + e^y \cos x)dx + e^y \sin x\, dy$
B: $x^2 dx + e^y \sin x\, dy$
C: $2xdx + e^y \cos x\, dx$
D: $2xdy + e^y \sin x\, dy$
E: None of the above

- $z = \frac{5x^2}{y-1} + 1$

A: $\frac{5x^2}{y-1} dx - \frac{5x^2}{(y-1)^2} dy$
B: $\frac{10x}{y-1} dx - \frac{5x^2}{y-1} dy$
C: $\frac{10x}{y-1} dx + \frac{5x^2}{(y-1)^2} dy$
D: $\frac{10x}{y-1} dx - \frac{5x^2}{(y-1)^2} dy$
E: None of the above

Exact differential

- A differential

$$dz = P(x,y)dx + Q(x,y)dy$$

is an exact differential if there exists a function $f(x,y)$ such that $P(x,y) = \frac{\partial f}{\partial x}$ and $Q(x,y) = \frac{\partial f}{\partial y}$. Then $z = f(x,y)$.

- Recall that for most nice functions $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$.
- Therefore, quick way to see if a differential is exact is to check if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Example

Solving exact differential equations

Integrating factors

- Sometimes, we can find an "integrating factor" $I(x)$ to multiply by both sides of an inexact ODE to make it an exact ODE.

Integrating factors

- Fact 1: every first-order ODE can be turned into an exact differential using an integrating factor.
- Fact 2: there is NO systematic way of guessing integrating factors for general ODEs.

- In MATA35, we will not expect you to use integrating factors outside of a few special cases where the integrating factors are known.

Integrating Factor for linear 1st-order ODE

- If you rewrite a linear 1st –order ODE in the following form:

$$y' + p(x)y = q(x)$$

  which is equivalent to

$$dy + dx[p(x)y] = q(x)$$

- Then the integrating factor is

$$e^{\int p(x)dx}$$

General solution for 1st-order ODE

- If you rewrite a linear 1st –order ODE in the following form:

$$y' + p(x)y = q(x)$$

- The general solution can be found by:
  - Determining the integrating factor $I(x) = e^{\int p(x)dx}$
  - Multiply both sides by $I(x)$: $y' \cdot I(x) + p(x)y \cdot I(x) = q(x) \cdot I(x)$
  - Multiply both sides by dx: $dy \cdot I(x) + p(x)y \cdot I(x)dx = q(x)I(x)dx$
  - The left hand side is the total differential $d[I(x)y]$
  - So we can integrate both sides to get $I(x)\mathrm{y} = \int q(x)I(x)dx$
  - Then $y = \frac{1}{I(x)}\left[\int q(x)I(x)dx + C\right]$
- In a single, ugly, long equation:

$$y(x) = e^{-\int p(x)dx}\left[\int e^{\int p(x)dx}dx + C\right]$$

Try it out

- $y' - 3x^2y = x^2$

A: $-\frac{1}{3} + e^{x^3} + C$
B: $-\frac{1}{3}e^{x^3} + C$
C: $-\frac{1}{3}e^{x^3+C}$
D: $-\frac{1}{3} + Ce^{x^3}$
E: None of the above