Linear 1st order ODEs and Integrating Factors Lecture 7d – 2021-07-07

MAT A35 – Summer 2021 – UTSC

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Existence-Uniqueness Theorem

• Consider the 1st order linear ODE initial value problem

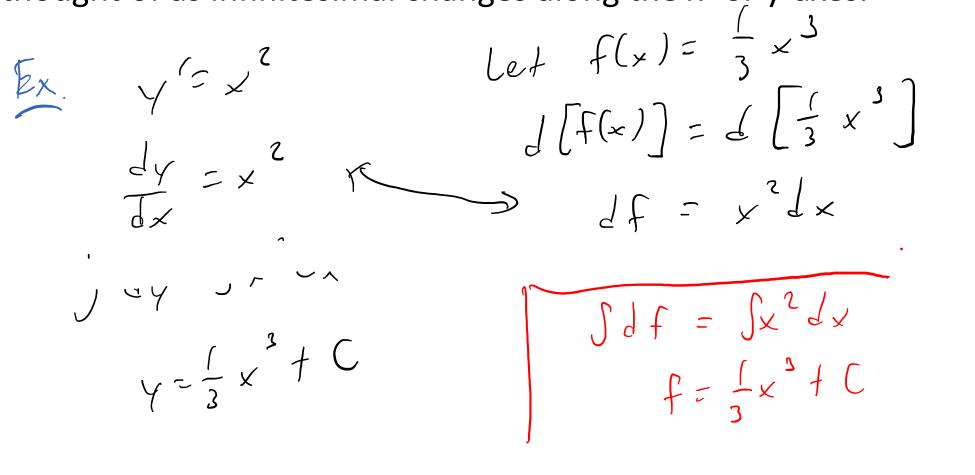
$$y' + p(x)y = q(x),$$
 $y(x_0) = y_0$

If p and q are continuous functions on an interval I containing x₀, then there exists a unique solution to the IVP for every point in I.

 In a more theoretical ordinary differential equations class, a lot of time is spent on proving various existence theorems, uniqueness theorems, and existence-uniqueness theorems.

Differentials

• Differentials dx and dy are the intuition behind $\frac{dy}{dx}$, and can be thought of as infinitesimal changes along the x- or y-axes.



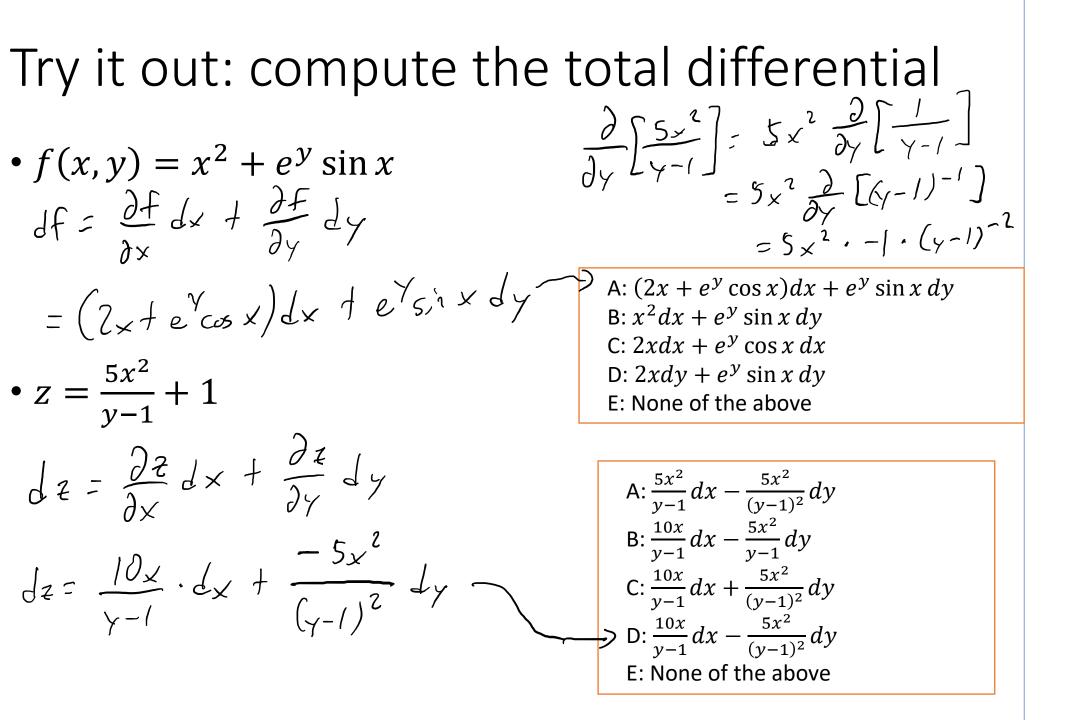
Differentials of multi-variable functions

- Let z = f(x, y) be a function of both x and y. Recall that the gradient $\nabla f = \begin{bmatrix} \partial f & \partial f \end{bmatrix}$. • Recall that the gradient $\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}$ gives the partial derivative in the $u = \begin{bmatrix} u_x \\ u_y \end{bmatrix}$ direction by $\nabla f \cdot u$.
- We define the total differential of z by

$$dz = f_x(x, y)dx + f_y(x, y)dy = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}\frac{dy}{dy}$$

• i.e.
$$dz = \nabla f \cdot \begin{bmatrix} ax \\ dy \end{bmatrix}$$
, where ∇f is the **second ar**.

Example of total differential Z=x 2+2xy +44 Ex. $dz = (2x + 2y) dx + (2x + 4y^{3}) dy$ dz = Zxdx + Zydx + Zxdy + 4y³dy $f(x,y) = sin 2x + y^2 e^{x}$ Ex, df = 2 cos 2x dx + y²e^xdx + 2ye^xdy $z = x^2 y^2$ EX. $d_{z} = \frac{\partial}{\partial x} \left[x^{2} y^{2} \right] d_{x} + \frac{\partial}{\partial y} \left[x^{2} y^{2} \right] d_{y} \mathbf{m}$ $= 2 \times y^2 d \times + 2 \times^2 y d y$



Reversing a total differential

•
$$dz = (2x + 2y)dx + (2x + 4y^3)dy$$

• Solve for $\frac{\partial z}{\partial x} = 2x + 2y$ and $\frac{\partial z}{\partial y} = 2x + 4y^3$
 $\int Y = \int (2x + 2y)dx \int Z = \int (2x + 4y^3)dy \int Z = \int (2x + 4y^3)dy \int Z = \int (2x + 4y^3)dy \int Z = 2xy + y^4 + G(x)$
 $z = x^2 + 2xy + F(y) + 0$
 $z = x^2 + 2xy + F(y) + 0$
 $z = x^2 + 2xy + y^4 + G(x)$

Try it out: Find z such that

•
$$dz = (2xy \cdot e^{x^2y})dx + (x^2 \cdot e^{x^2y} + 5)dy$$

 $z = \int 2xy e^{x^2y}dx$
 $z = \int (x^2 e^{x^2y} + 5)dy$
 $z = e^{x^2y} + F(y)$
 $z = e^{x^2y} + 5y + G(x)$
 $z = e^{x^2y} + 5y$
 $z = e^{x^2y} + 5y$

A:
$$e^{x^2y} + 5y$$

B: $x^2ye^{x^2y} + 5xy$
C: $x^2ye^{x^2y} + 5y$
D: $e^{2xy} + 5y$
E: None of the above

Sometimes, reversing fails

• $dz = y^2 dx + x^2 dy$ \downarrow $z = \int_{Y}^{2} d_{X}$ $z = \int_{X}^{2} d_{Y}$ $z = x^2 + F(y)$ $z = x^2 + F(y)$ $z = x^2 + F(y)$

dz is NOT EXACT

Exact differential

• A differential

$$dz = P(x, y)dx + Q(x, y)dy$$

is an exact differential if there exists a function f(x, y) such that $P(x, y) = \frac{\partial f}{\partial x}$ and $Q(x, y) = \frac{\partial f}{\partial y}$. Then z = f(x, y).

- In other words, an exact differential is any differential that is the total differential of some function.
- An inexact differential is a differential we write down that is not the total differential of any function.

Differential test for exactness

- One way to test for exactness is to try to reverse the differential; this will always work, but involves a lot of integration.
- There is a faster test that only involves differentiation.
- Recall that for most nice functions $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$.
- Therefore, quick way to see if a differential dz = P(x, y)dx + Q(x, y)dy

is exact is to check if $\frac{\partial P}{\partial v} = \frac{\partial Q}{\partial x}$

Example $d = \frac{10x}{y-1} d = \frac{5x^2}{(y-1)^2} d = \frac{5x^2}{$ Ex. Q(x,y)P(x,y) $\frac{\partial Q}{\partial y} = \frac{-10x}{(y-1)^2}$ $\frac{\partial Y}{\partial y} = \frac{-10x}{(y-1)^2}$ Exact $dz = (5x^2 + 1) dx + xy^2 dy$ EL, Q(x,y) $P(x, \gamma)$ Nut exict $\frac{\partial Q}{\partial \chi} = \gamma^2$ 2P = 0

Try it out: exact or inexact?

• dz = xdx + ydy $\frac{\partial}{\partial y} [x] = 0$ $\frac{\partial}{\partial x} [y] = 0$

Exact.
$$z = \frac{1}{2}x^2 + \frac{1}{2}y$$

- dz = ydx + xdy $\frac{\partial}{\partial y} [y] = 1 = \frac{\partial}{\partial x} [x] = 1 = \frac{\partial}{\partial x} [x]$
- $dz = xdx + y^2dy$ $\frac{\partial}{\partial \gamma} \left[\frac{\partial}{\partial \gamma} \right] = 0$ $\frac{\partial}{\partial z} \left[\frac{\partial}{\partial \gamma} \right] = 0$

Exact.
$$Z = \frac{1}{3} \times \frac{1$$

A: exact B: inexact C: both exact and inexact D: ??? E: None of the above

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- $dz = y^2 dx + x dy$ $\frac{\partial}{\partial y} [y^2] = 2y \neq \frac{\partial}{\partial x} [x] = 1$ Inexact.
- dz = (x + y)dx + (x + y)dy $\frac{\partial}{\partial y} [x + y] = 1$ $\frac{\partial}{\partial x} [x + y] = 1$ $\frac{1}{2} [x + y] = 1$ $\frac{1}{2} [x + y]^2$

Solving exact differential equations

 $\frac{dy}{dx} = \frac{-2xy}{x^2+1}$

=) exact

Jdy=xtC

Jdy = ytC1

 $\int Z_{xy} dx = x^2 y + F(y)$ $\int (x^2 + i) dy = x^2 y + y + G(x)$ (x2+1)dy = -2xydx $=) f(x,y) = x^{2}y + y f = x^{2}y + y$ $2xydx + (x^2+1)dy = 0$ $df = 2xy dx + (x^2 + 1) dy = 0$ $\frac{\partial}{\partial y} [2_{xy}] = 2_{x} \frac{\partial}{\partial x} [x^{2} + 1] = 2_{x}$ =) 1t=0 $\int Jf = ff C = \int O = 0$ $=) \frac{2}{x y + y + C} = 0$ =) $(x^{2}+1)y^{-1}$ =) $\gamma^{2} \sqrt{2} \frac{1}{\sqrt{2}}$

Integrating factors

• Sometimes, we can find an "integrating factor" I(x) to multiply by both sides of an inexact ODE to make it an exact ODE.

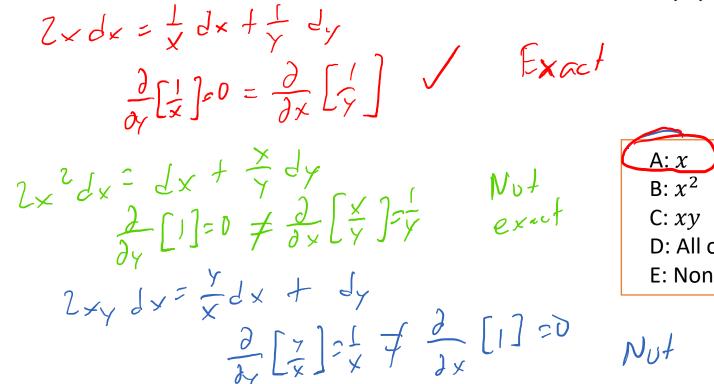
 $E_{x_1} = x_1 + 2y_2 = 5x^3$ Let I(x) = x Multiply by I(x) $x \cdot \frac{dy}{dx} + 2y = 5x^{2}$ =) e xactLet $f(x,y) = x^2 y$ =) $\int df = \int 5x^4 dx$ $f = x^5 f C$ $\frac{\partial}{\partial y} [2y] = 2 \neq \frac{\partial}{\partial x} [x] = 1$ NOT EXACT

Try it out:

Given the inexact differential equation

$$2dx = \frac{1}{x^2}dx + \frac{1}{xy}dy$$

• Which of the following is an integrating factor I(x)?



D: All of the above

E: None of the above

exact

Integrating factors

- Fact 1: every first-order ODE can be turned into an exact differential using an integrating factor.
- Fact 2: there is NO systematic way of guessing integrating factors for general ODEs.
- In MATA35, we will not expect you to use integrating factors outside of a few special cases where the integrating factors are known, or where we give you a hint.

Integrating Factor for linear 1st-order ODE

• If you rewrite a linear 1st –order ODE in the following form:

$$y' + p(x)y = q(x)$$

which is equivalent to

$$dy + dx[p(x)y] = q(x)dx$$

• Then the integrating factor is

Then the integrating factor is $e^{\int p(x)dx} = e^{\int \frac{2}{x} dx} = e^{\ln x^{2}}$ $f(x) = x^{2}$ $f(x) = x^{2}$

General solution for 1st-order ODE

- If you rewrite a linear 1st –order ODE in the following form: y' + p(x)y = q(x)
- The general solution can be found by:
 - Determining the integrating factor $I(x) = e^{\int p(x)dx}$
 - Multiply both sides by $I(x): y' \cdot I(x) + p(x)y \cdot I(x) = q(x) \cdot I(x)$
 - Multiply both sides by dx: $dy \cdot I(x) + p(x)y \cdot I(x)dx = q(x)I(x)dx$
 - The left hand side is the total differential d[I(x)y]
 - So we can integrate both sides to get $I(x)y = \int q(x)I(x)dx$

• Then
$$y = \frac{1}{I(x)} \left[\int q(x)I(x)dx + C \right]$$

• In a single, ugly, long equation:

$$y(x) = e^{-\int p(x)dx} \left[\int e^{\int p(x)dx} dx + C \right]$$

