Linear 1st order ODEs and Integrating Factors Lecture 7d – 2021-07-07

MAT A35 – Summer 2021 – UTSC

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Existence-Uniqueness Theorem

• Consider the 1st order linear ODE initial value problem

$$y' + p(x)y = q(x),$$
 $y(x_0) = y_0$

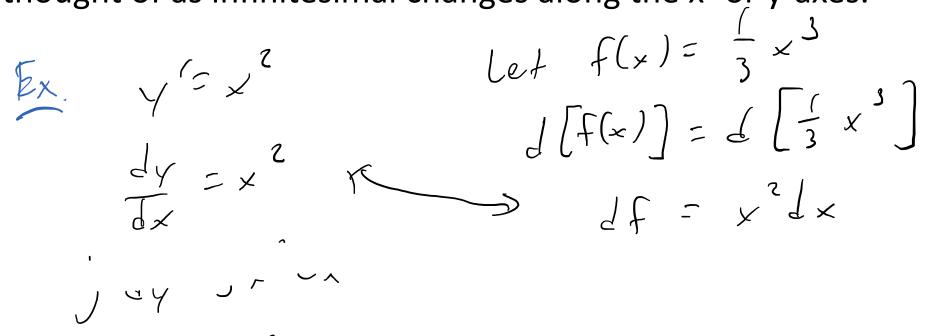
If p and q are continuous functions on an interval I containing x₀, then there exists a unique solution to the IVP for every point in I.

 In a more theoretical ordinary differential equations class, a lot of time is spent on proving various existence theorems, uniqueness theorems, and existence-uniqueness theorems.

Differentials

 $y = \frac{1}{2} x^3 + C$

• Differentials dx and dy are the intuition behind $\frac{dy}{dx}$, and can be thought of as infinitesimal changes along the x- or y-axes.



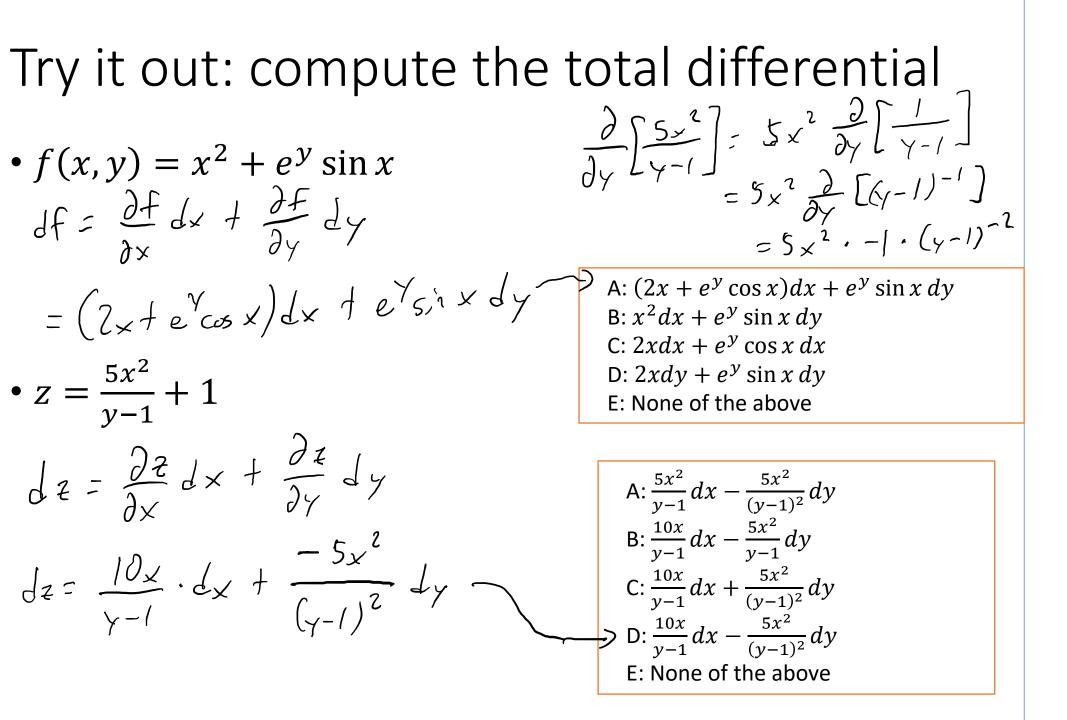
Differentials of multi-variable functions

- Let z = f(x, y) be a function of both x and y. Recall that the gradient $\nabla f = \begin{bmatrix} \partial f & \partial f \end{bmatrix}$. • Recall that the gradient $\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}$ gives the partial derivative in the $u = \begin{bmatrix} u_x \\ u_y \end{bmatrix}$ direction by $\nabla f \cdot u$.
- We define the total differential of z by

$$dz = f_x(x, y)dx + f_y(x, y)dy = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}\frac{dy}{dy}$$

• i.e.
$$dz = \nabla f \cdot \begin{bmatrix} ax \\ dy \end{bmatrix}$$
, where ∇f is the **second ar**.

Example of total differential Z=x 2+2xy +44 Ex. $dz = (2x + 2y) dx + (2x + 4y^{3}) dy$ dz = Zxdx + Zydx + Zxdy + 4y³dy $f(x,y) = sin 2x + y^2 e^{x}$ Ex, df = 2 cos 2x dx + y²e^xdx + 2ye^xdy $z = x^2 y^2$ EX. $d_{z} = \frac{\partial}{\partial x} \left[x^{2} y^{2} \right] d_{x} + \frac{\partial}{\partial y} \left[x^{2} y^{2} \right] d_{y} \mathbf{m}$ $= 2 \times y^2 d \times + 2 \times^2 y d y$



Reversing a total differential

•
$$dz = (2x + 2y)dx + (2x + 4y^3)dy$$

• Solve for $\frac{\partial z}{\partial x} = 2x + 2y$ and $\frac{\partial z}{\partial y} = 2x + 4y^3$

Try it out: Find z such that

•
$$dz = (2xy \cdot e^{x^2y})dx + (x^2 \cdot e^{x^2y} + 5)dy$$

A:
$$e^{x^2y} + 5y$$

B: $x^2ye^{x^2y} + 5xy$
C: $x^2ye^{x^2y} + 5y$
D: $e^{2xy} + 5y$
E: None of the above

Sometimes, reversing fails

• $dz = y^2 dx + x^2 dy$

Exact differential

• A differential

$$dz = P(x, y)dx + Q(x, y)dy$$

is an exact differential if there exists a function f(x, y) such that $P(x, y) = \frac{\partial f}{\partial x}$ and $Q(x, y) = \frac{\partial f}{\partial y}$. Then z = f(x, y).

- In other words, an exact differential is any differential that is the total differential of some function.
- An inexact differential is a differential we write down that is not the total differential of any function.

Differential test for exactness

- One way to test for exactness is to try to reverse the differential; this will always work, but involves a lot of integration.
- There is a faster test that only involves differentiation.
- Recall that for most nice functions $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$.
- Therefore, quick way to see if a differential dz = P(x, y)dx + Q(x, y)dyis event is to check if $\frac{\partial P}{\partial Q} = \frac{\partial Q}{\partial Q}$

is exact is to check if
$$\frac{1}{\partial y} = \frac{1}{\partial x}$$

Example

Try it out: exact or inexact?

- dz = xdx + ydy
- dz = ydx + xdy
- $dz = xdx + y^2dy$
- $dz = y^2 dx + x dy$
- dz = (x + y)dx + (x + y)dy

A: exact B: inexact C: both exact and inexact D: ??? E: None of the above

Solving exact differential equations

Integrating factors

• Sometimes, we can find an "integrating factor" I(x) to multiply by both sides of an inexact ODE to make it an exact ODE.

Try it out:

• Given the inexact differential equation

$$2dx = \frac{1}{x^2}dx + \frac{1}{xy}dy$$

• Which of the following is an integrating factor I(x)?

A: *x* B: *x*²

C: *xy*

D: All of the above

E: None of the above

Integrating factors

- Fact 1: every first-order ODE can be turned into an exact differential using an integrating factor.
- Fact 2: there is NO systematic way of guessing integrating factors for general ODEs.
- In MATA35, we will not expect you to use integrating factors outside of a few special cases where the integrating factors are known, or where we give you a hint.

Integrating Factor for linear 1st-order ODE

• If you rewrite a linear 1st –order ODE in the following form:

$$y' + p(x)y = q(x)$$

which is equivalent to

$$dy + dx[p(x)y] = q(x)$$

• Then the integrating factor is $e^{\int p(x)dx}$

General solution for 1st-order ODE

- If you rewrite a linear 1st –order ODE in the following form: y' + p(x)y = q(x)
- The general solution can be found by:
 - Determining the integrating factor $I(x) = e^{\int p(x)dx}$
 - Multiply both sides by $I(x): y' \cdot I(x) + p(x)y \cdot I(x) = q(x) \cdot I(x)$
 - Multiply both sides by dx: $dy \cdot I(x) + p(x)y \cdot I(x)dx = q(x)I(x)dx$
 - The left hand side is the total differential d[I(x)y]
 - So we can integrate both sides to get $I(x)y = \int q(x)I(x)dx$

• Then
$$y = \frac{1}{I(x)} \left[\int q(x)I(x)dx + C \right]$$

• In a single, ugly, long equation:

$$y(x) = e^{-\int p(x)dx} \left[\int e^{\int p(x)dx} dx + C \right]$$

Try it out

•
$$y' - 3x^2y = x^2$$

A:
$$-\frac{1}{3} + e^{x^3} + C$$

B: $-\frac{1}{3}e^{x^3} + C$
C: $-\frac{1}{3}e^{x^3 + C}$
D: $-\frac{1}{3} + Ce^{x^3}$
E: None of the above