Direction fields, autonomous ODEs, and the phase line
Lecture 7e – 2021-07-07

MAT A35 – Summer 2021 – UTSC

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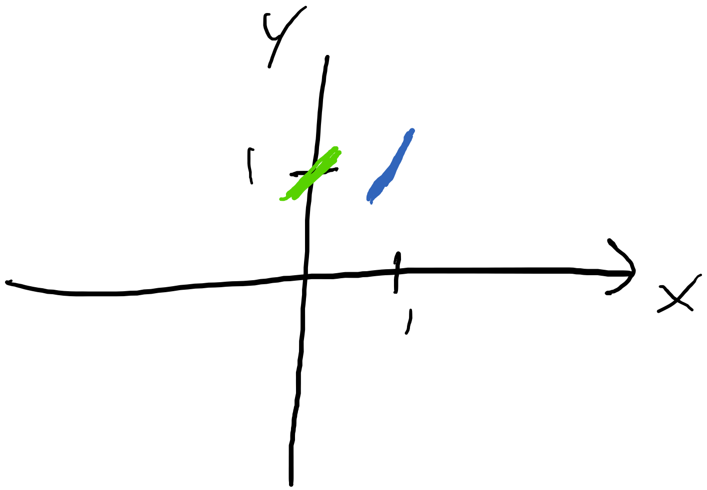
1^{st}-order ODEs and slopes of solutions

- We can write a 1^{st}-order ODE as $y' = f(x, y)$
- Recall that the derivative can be thought of as the slope of a solution.

Ex. $y' = x + y$

A solution passing through $y(\overset{x}{1}) = \overset{y}{1}$ has slope $y' = 2$ at the point $(1, 1)$

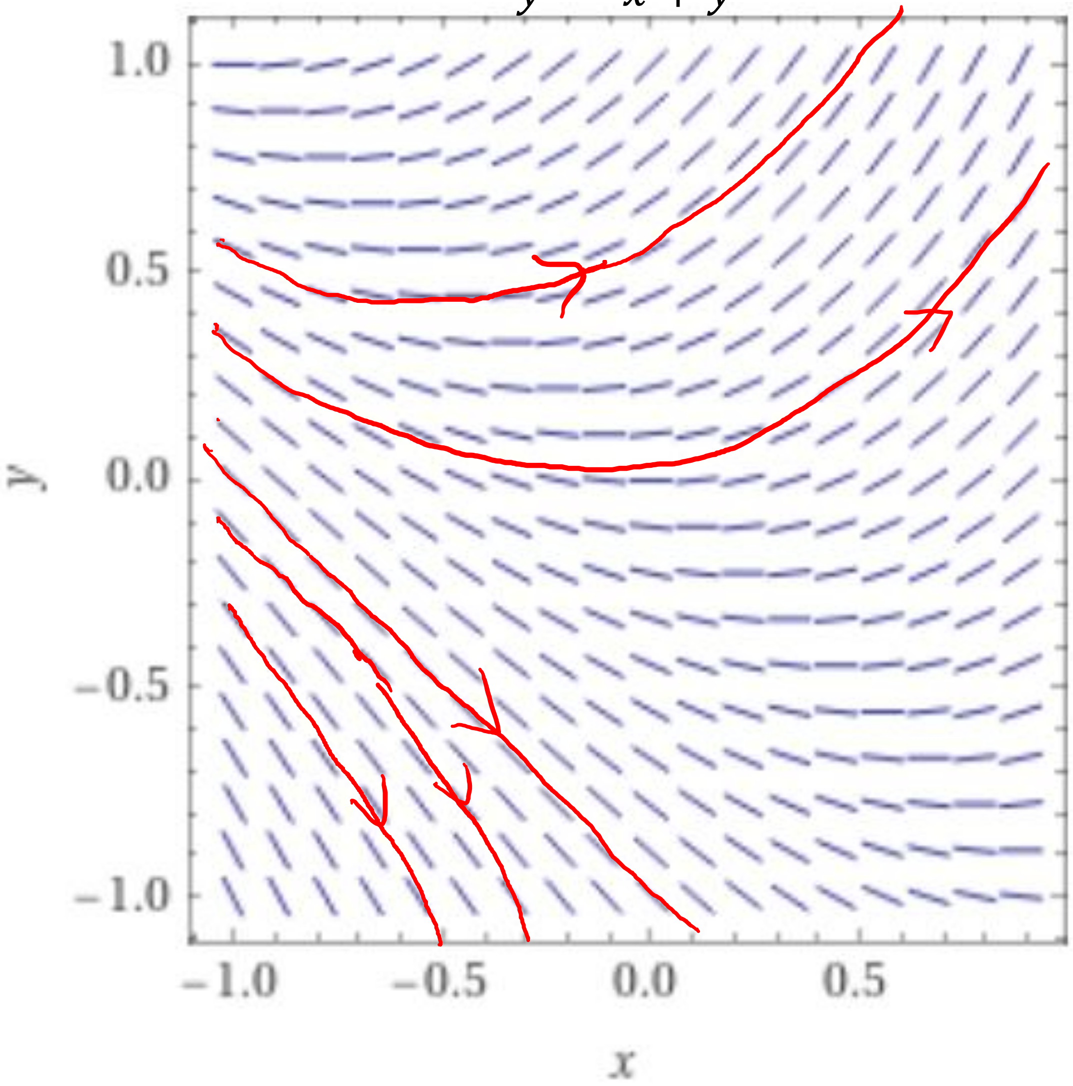


A solution passing through $y(0) = 1$ has slope $y' = 1$

Direction field

- A direction field graphs out the slopes of all solutions going through a point.
- We can visualize different solutions by drawing trajectory curves that are always tangent to the direction field.

$y' = x + y$



https://www.wolframalpha.com/input/?i=slope+field+of+y%27%3Dx%2By

Autonomous ODEs

- Recall that an autonomous ODE is one that does not have an explicit dependence on the independent variable (e.g. time).
- A first-order autonomous ODE can be rewritten in the form:

$$y' = f(y)$$

Ex. $y'' + y' + y = 0$ – 2nd order autonomous

Ex. $(y')^2 - \sin y - 1 = 0$ – 1st order autonomous

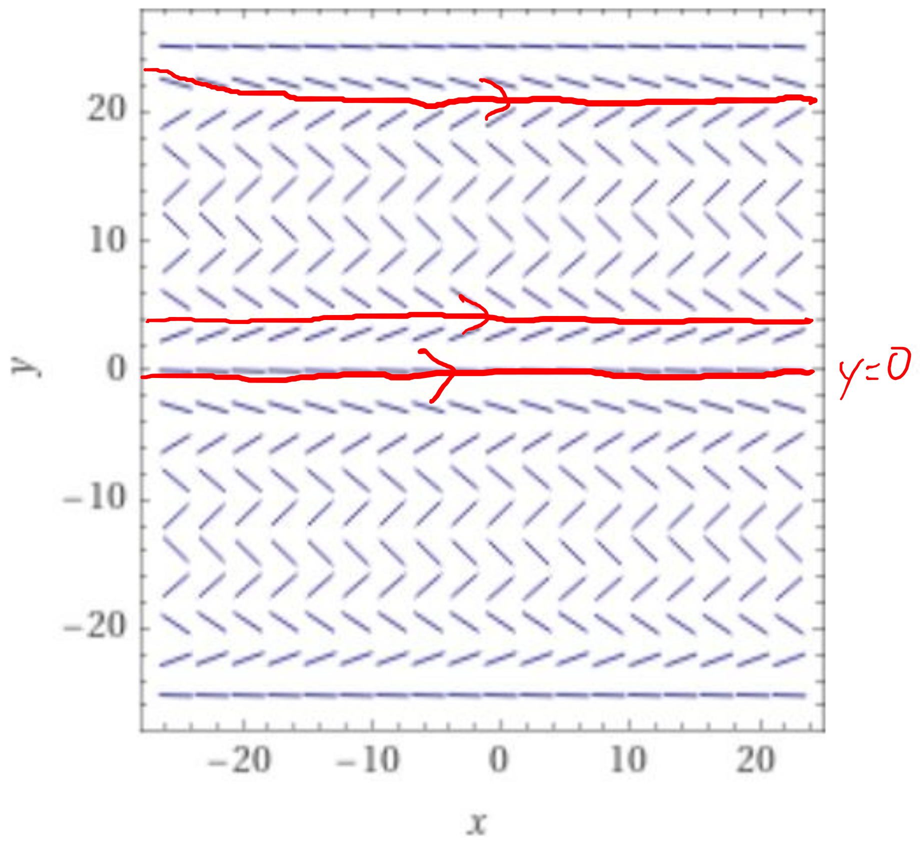
$\Rightarrow (y')^2 = \sin y + 1$

$y' = \sqrt{\sin y + 1}$ – standard form

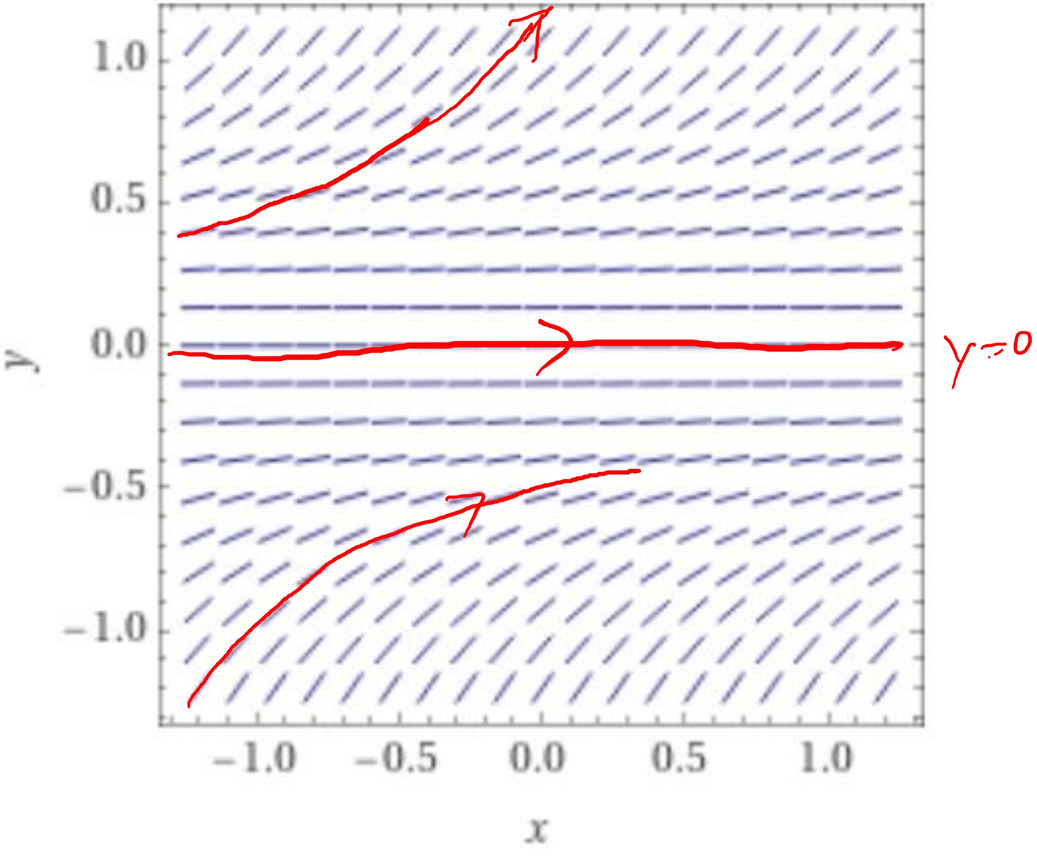
Direction fields of autonomous ODEs

- Notice that if $y' = f(y)$, then the slope has no x-dependence.

$y' = \sin y$



$y' = y^2$



Equilibrium values

- An equilibrium value of the autonomous ODE $y' = f(y)$ is a constant solution $y = c$.
- We can solve for equilibrium values by setting $y' = 0$.

Ex. $y' = y^2 = 0$

$\Rightarrow y = 0$ is the only equilibrium solution

Ex. $y' = \sin y = 0$

$\Rightarrow y = k\pi$, k is any integer,

are all equilibria

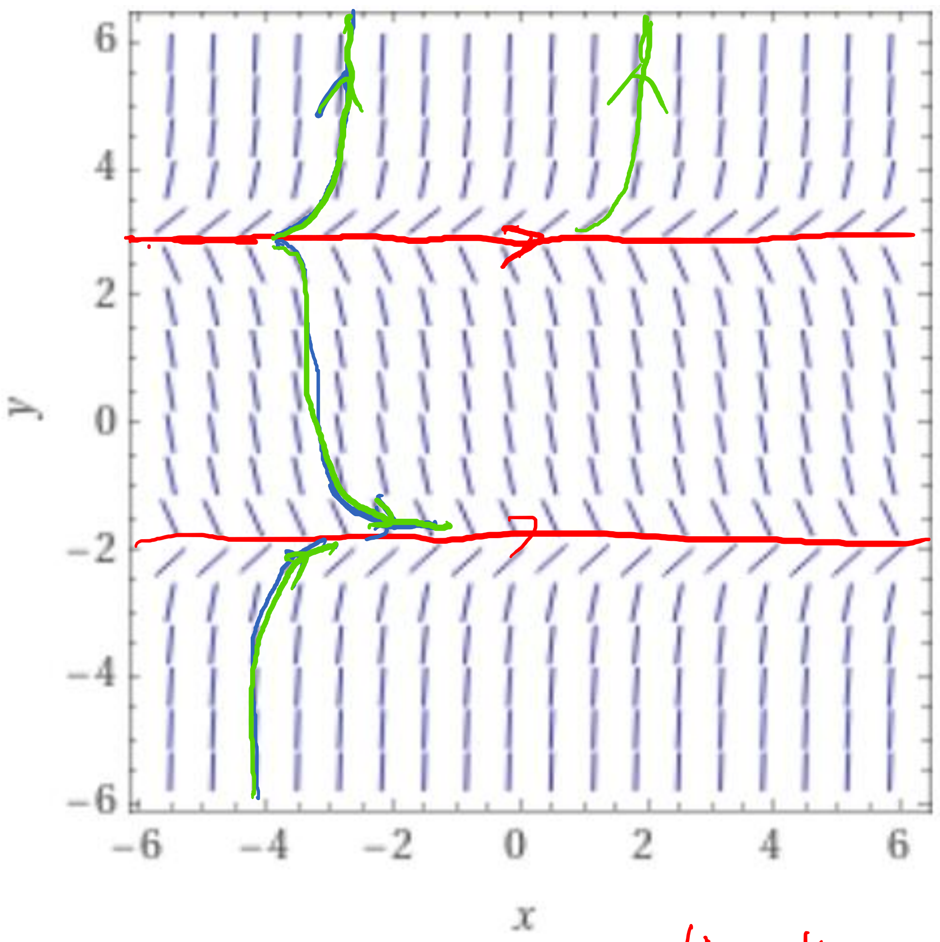
Try it out: find the equilibrium values

- $y' = y^2 - y - 6$

$y^2 - y - 6 = 0$

$(y-3)(y+2) = 0$

$y = 3, -2$



unstable

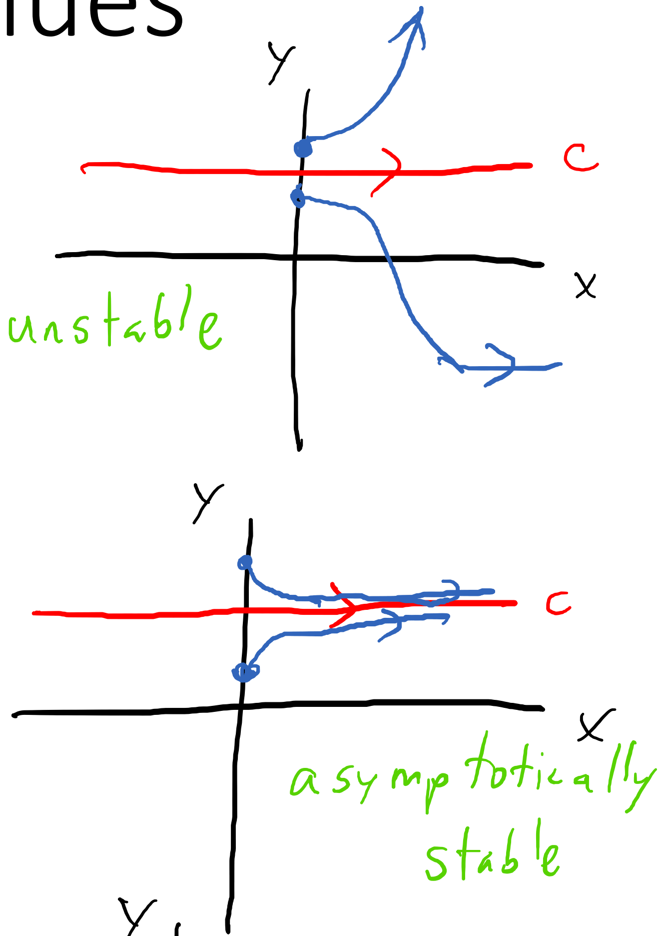
asymptotically stable

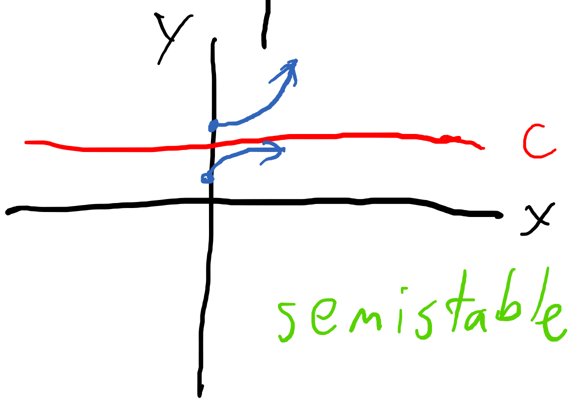
$\longrightarrow$ direction of time

A: -2
B: 3
C: All of the above
D: ???
E: None of the above

Stability of equilibrium values

- Consider an equilibrium value c of $y' = f(y)$, and an initial value $y(0) = y_0$, where $y_0 \approx c$, but $y_0 \neq c$. Then c is
  - Unstable if y diverges from c as time $x \to \infty$
  - Asymptotically stable if $y(x) \to c$ as $x \to \infty$
  - Semi-stable if as $x \to \infty$, $y(x)$ goes to c on one side, but diverges on the other side.

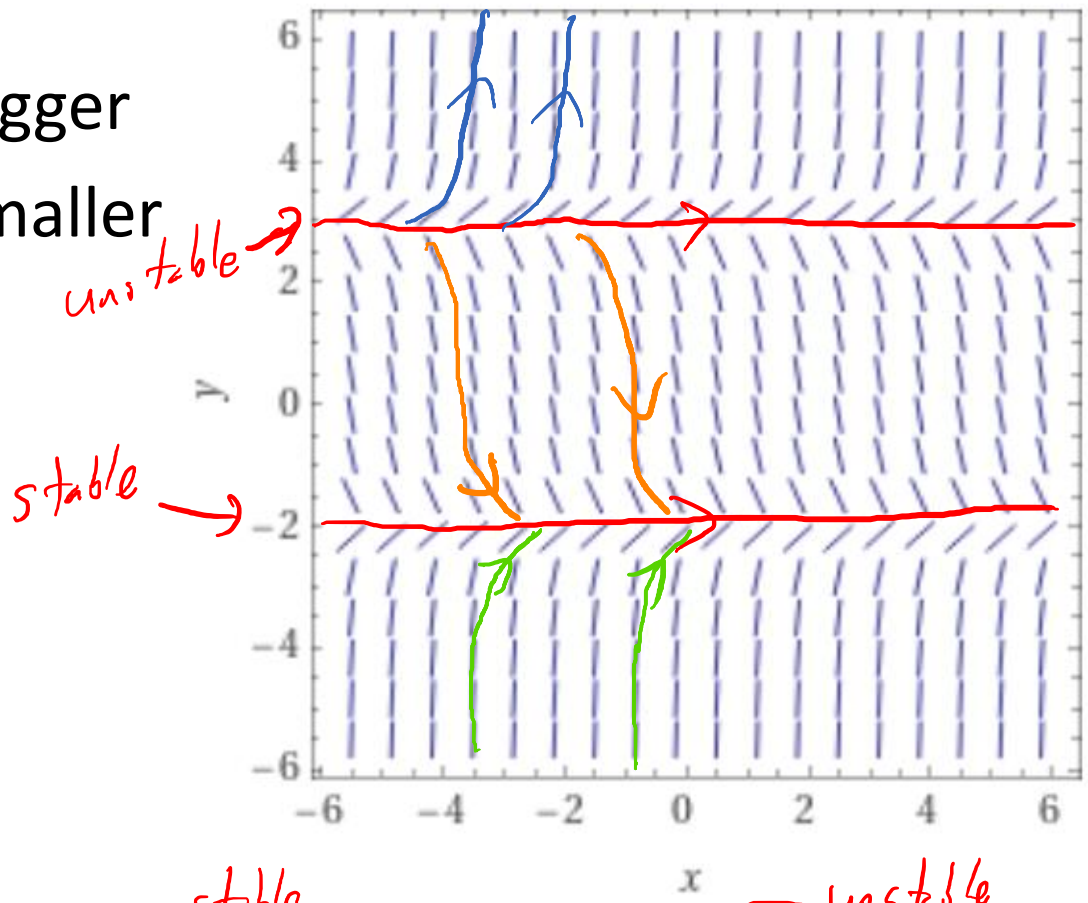




Determining stability using sign of y'

- $y' = 0$ at equilibrium.
- $y' > 0$ implies $y(x)$ gets bigger
- $y' < 0$ implies $y(x)$ gets smaller

$y' = y^2 - y - 6$



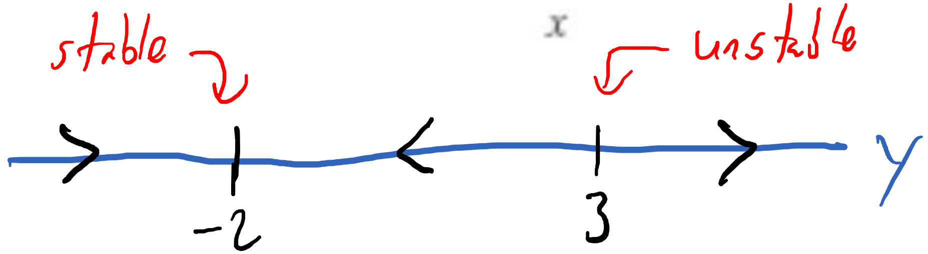
Ex. $y' = y^2 - y - 6$

$y' = (y-3)(y+2)$

$y < -2$: $y' = (-)(-) = (+)$

$-2 < y < 3$, $y' = (-)(+) = (-)$

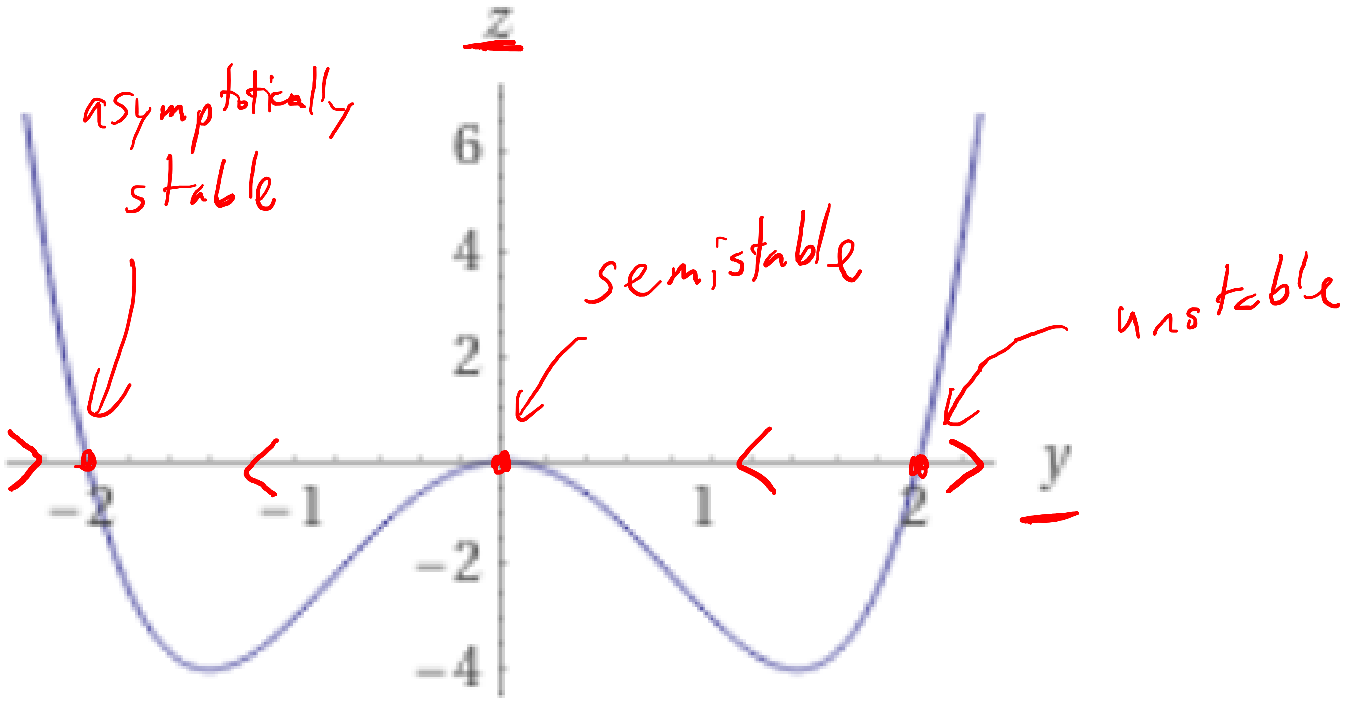
$3 < y$ $y' = (+)(+) = (+)$



Phase line

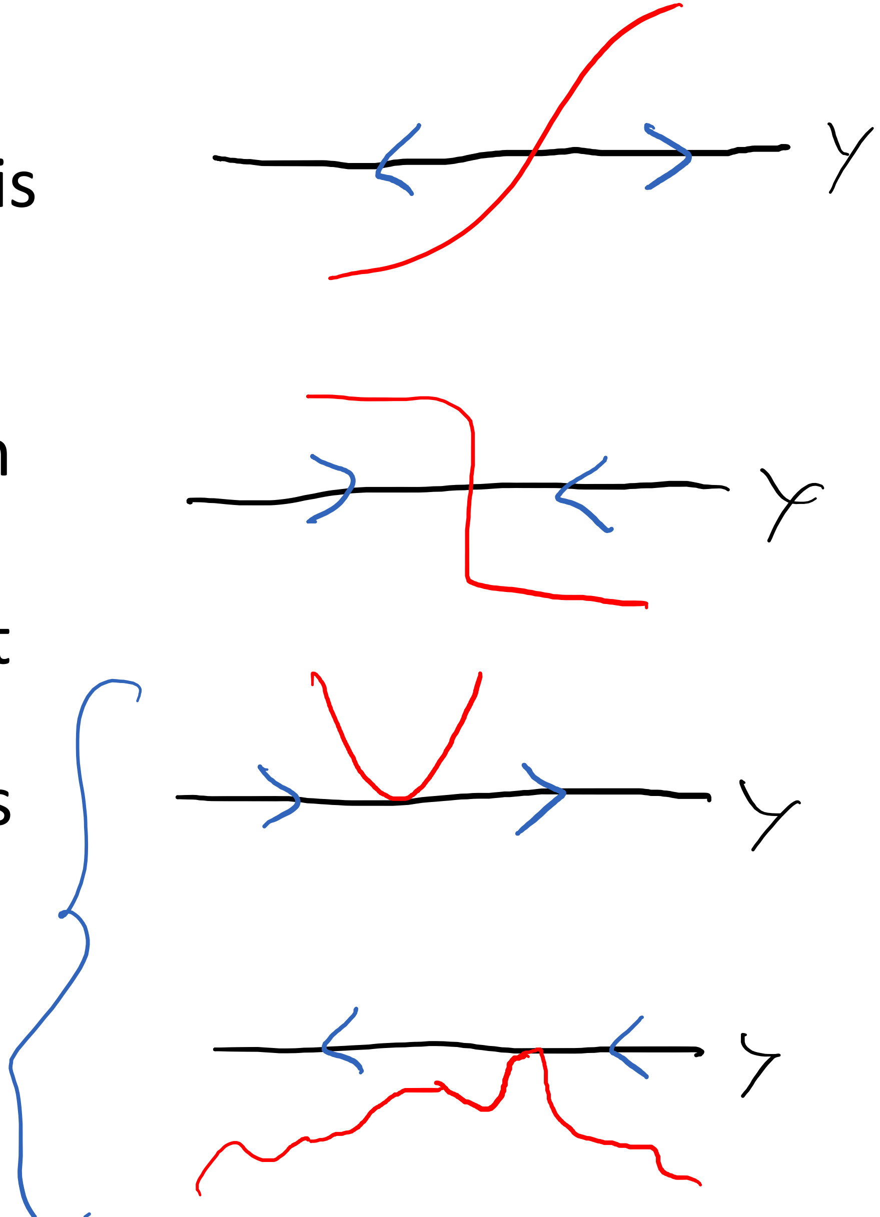
- Draw arrows along the y-axis depending on if $z = f(y) = y'$ is positive or negative.

$$y' = y^4 - 4y^2$$



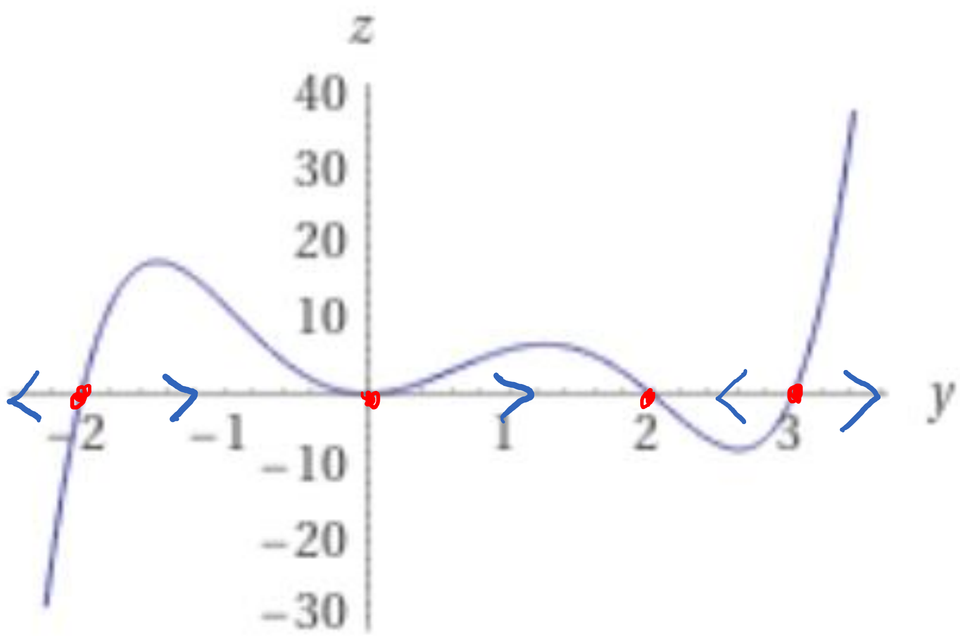
Phase line stability of $y' = f(y)$

- If $z = f(y)$ crosses the y-axis at $f(c) = 0$ going upward, then c is unstable.
- If $z = f(y)$ crosses the y-axis at $f(c) = 0$ going downward, then c is asymptotically stable.
- If $z = f(y)$ touches the y-axis at $f(c) = 0$, but remains on the same side of the y-axis, then c is semi-stable.



Try it out

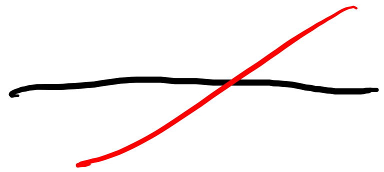
- $y' = y^5 - 3y^4 - 4y^3 + 12y^2$
- Classify the stability of the following equilibria:
- $y = -2$ unstable
- $y = 0$ semistable
- $y = 2$ asymptotically stable
- $y = 3$ unstable



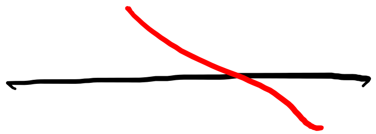
A: Asymptotically stable
B: Unstable
C: Semi-stable
D: ???
E: None of the above

Derivative test for stability

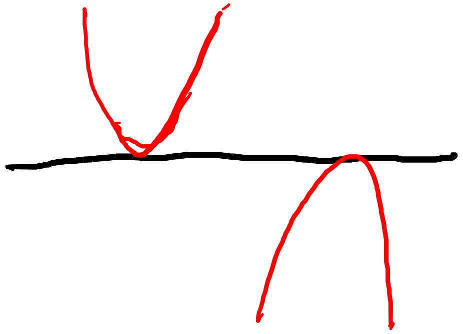
- Let $y' = f(y)$ have an equilibrium at $f(c) = 0$.
- If $\frac{df}{dy}(c) > 0$, then $y = c$ is unstable.



- If $\frac{df}{dy}(c) < 0$, then $y = c$ is asymptotically stable.



- If $\frac{df}{dy}(c) = 0$ and c is a local extremum (max or min) of $f(y)$, then $y = c$ is semi-stable



Example

- $y' = y^5 - 3y^4 - 4y^3 + 12y^2$ has equilibria -2, 0, 2, 3

$f(y) = y^5 - 3y^4 - 4y^3 + 12y^2$

$\frac{df}{dy} = 5y^4 - 12y^3 - 12y^2 + 24y$

$\frac{d^2f}{dy^2} = 20y^3 - 36y^2 - 24y + 24$

$\frac{df}{dy}(-2) = 80 + 96 - 48 - 48$

$= 80 > 0$

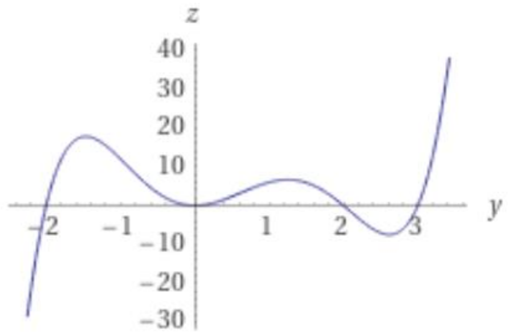
so -2 is unstable

$\frac{df}{dy}(0) = 0$ $\quad \frac{d^2f}{dy^2} = 24$, so 0 is a local min of $f(y)$

so 0 is semistable

$\frac{df}{dy}(2) = -16$, so 2 is stable

$\frac{df}{dy}(3) = 45$, so 3 is unstable



Try it out

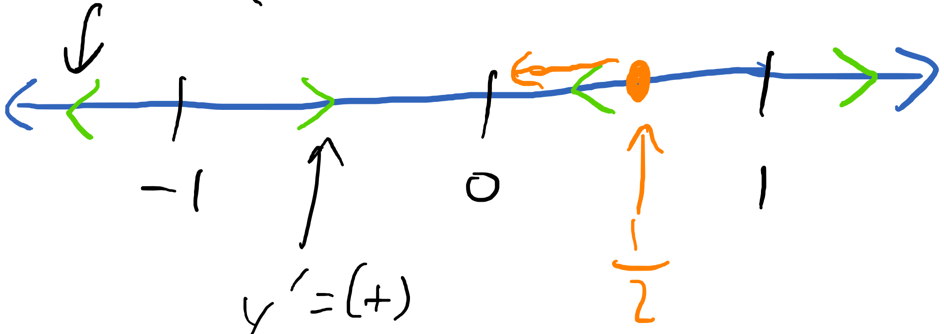
- Fat crystallization: Let $y(x)$ be the proportion of crystallizable milk fat in a sample after x hours, satisfying
$$y' = 8(y^5 - y)$$
- If you start with half of the fat as crystallizable, how much fat is crystallizable as time goes to ∞?

https://commons.wikimedia.org/wiki/File:Glass_of_milk.jpg

A: All of the fat
B: Half of the fat
C: None of the fat
D: ???
E: None of the above

$y < -1$

$y' = (-)(+)(-)(-)$
$= (-)$



$y' = 0 = 8(y^5 - y)$

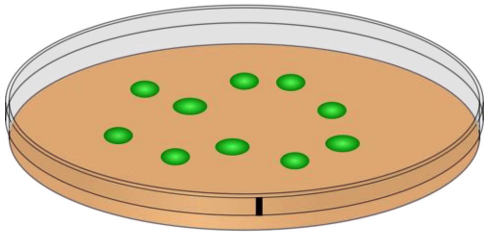
$0 = y(y^4 - 1)$

$0 = y(y^2+1)(y^2-1)$

$0 = y(y^2+1)(y+1)(y-1)$

Equilibria: $y = -1, 0, 1$

Logistic Growth Model



- Previously, we saw exponential growth $y' = ky$, $y(0) = y_0$, which had a solution $y(x) = y_0 e^{kx}$.
- In practice, this is unrealistic. For example, bacteria in a petri dish will initially grow almost exponentially, but then they'll use up all the available media.
- A better model is the logistic model, $y' = ky\left(1 - \frac{y}{L}\right)$, where the parameter L is the carrying capacity of the environment, and $k > 0$ is still the growth rate.

Stability of equilibria of logistic model

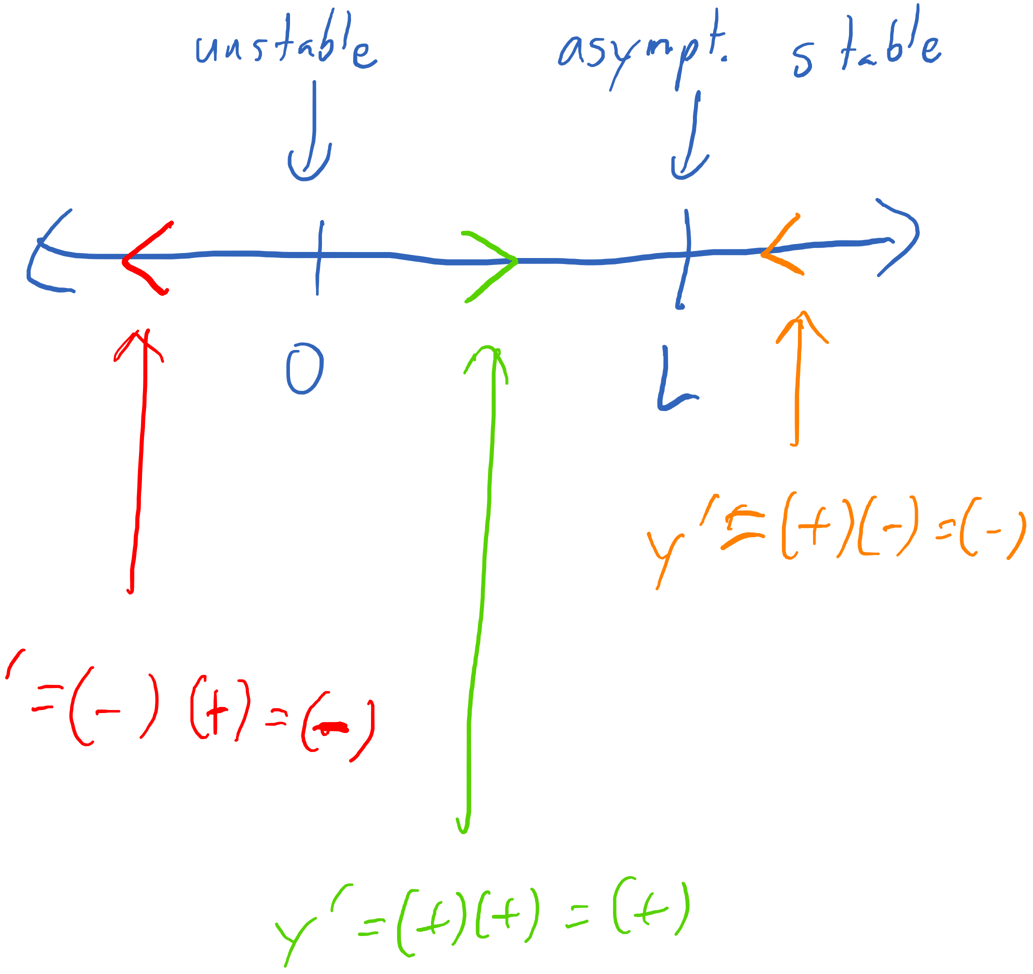
- $y' = ky\left(1 - \frac{y}{L}\right)$

$y' = 0 = ky\left(1 - \frac{y}{L}\right)$

$y = 0$ or $1 - \frac{y}{L} = 0$

$\Rightarrow y = L$

Two equilibria: 0 and L



Logistic model behavior

- $y' = ky\left(1 - \frac{y}{L}\right)$
- $y < 0$ is not physical, since we cannot have negative bacteria.
- When $0 < y < L$, $y' > 0$, so the number of bacteria increase, up to L.
- When $y > L$, $y' < 0$, so the number of bacteria decrease, down to L.
- On large time-scales, we therefore have L bacteria.