Compartmental Models
Lecture 8a – 2021-07-09

MAT A35 – Summer 2021 – UTSC

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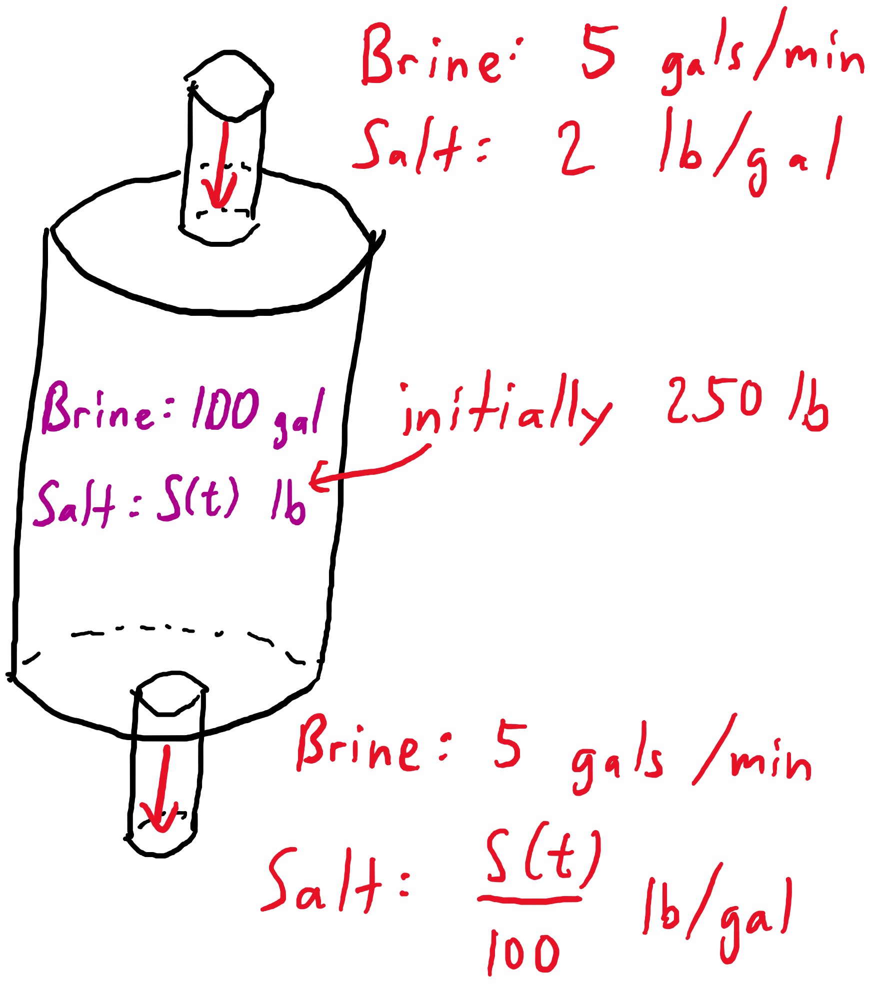
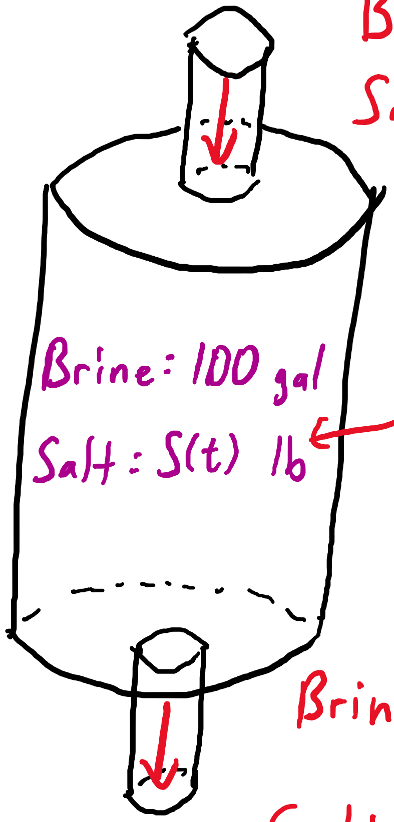
Mixing problem (Bittinger, pg 559, ex. 7)

- A tank contains 100 gallons of brine whose concentration is 2.5 lb of salt / gallon.
- Brine containing 2 lb of salt / gallon runs into the tank at a rate of 5 gallons / min
- The brine in the tank runs out at the same rate of 5 gallons / min.
- How does the amount of salt $S(t)$ in the tank change over time?



Rate of change in amount of salt



Brine: 5 gals/min
Salt: 2 lb/gal

} 10 lb/min of salt added to tank

initially 250 lb

$S(0) = 250$

$$\boxed{\frac{dS}{dt} = \dot{S} = 10 - \frac{S}{20}}$$ lb/min

Brine: 5 gals/min
Salt: $\frac{S(t)}{100}$ lb/gal

} $\frac{S(t)}{20}$ lb/min of salt added to tank

Solving for $\dot{S} = 10 - \frac{S}{20}$

$\dot{S} = \frac{dS}{dt}$

$$\dot{S} + \underbrace{\frac{1}{20}}_{p(t)} \cdot S = \underbrace{10}_{q(t)}$$

linear 1st-order ODE

$$dS + dt\left[\frac{1}{20}\cdot S\right] = 10\,dt$$

$$I(t) = e^{\int \frac{1}{20}dt} = e^{\frac{t}{20}}$$

(Integrating factor)

$$\int e^{\frac{t}{20}}\left[dS + dt\left(\frac{1}{20}\cdot S\right)\right] = \int 10e^{\frac{t}{20}}dt$$

$$e^{\frac{t}{20}}\cdot S = 200e^{t/20} + C$$

$$S = 200 + Ce^{-t/20}$$

(multiply by $e^{-t/20}$)

Initial value problem

- $S = 200 + Ce^{-\frac{t}{20}}$ and $S(0) = 250$

$250 = S(0) = 200 + Ce^{0} = 200 + C$

$C = 50$

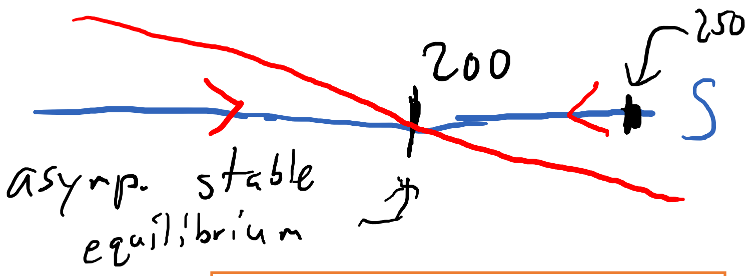
$S = 200 + 50e^{-t/20}$

Salt amount in tank is $200 + 50e^{-t/20}$ lbs

$\dot{S} = 10 - \frac{S}{20}$

$10 - \frac{S}{20} = 0$

$\Rightarrow S = 200$ equilibrium



Amount of salt in tank on long time-scales?

A: 50 lb

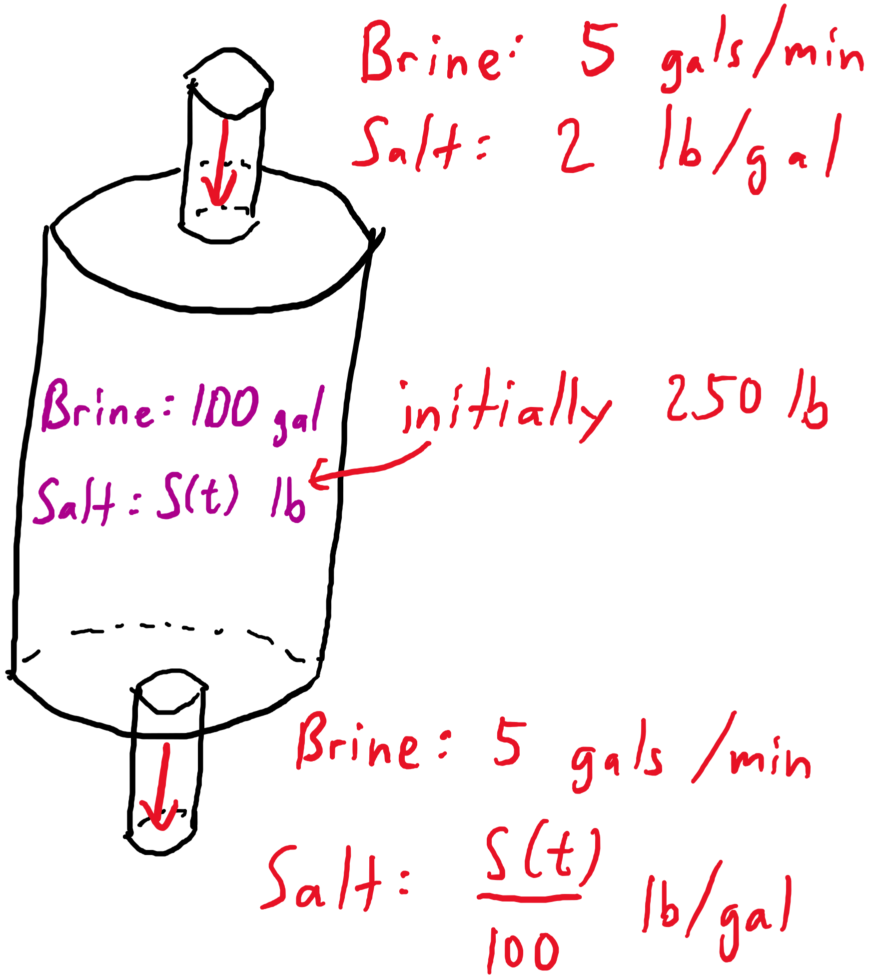
B: 50 lb / min

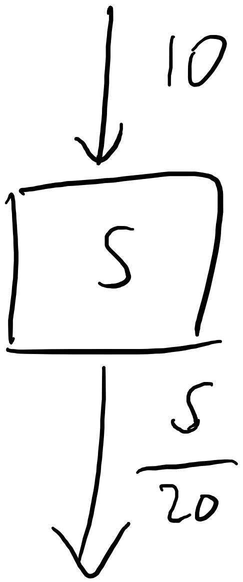
C: 200 lb

D: 200 lb / min

E: None of the above

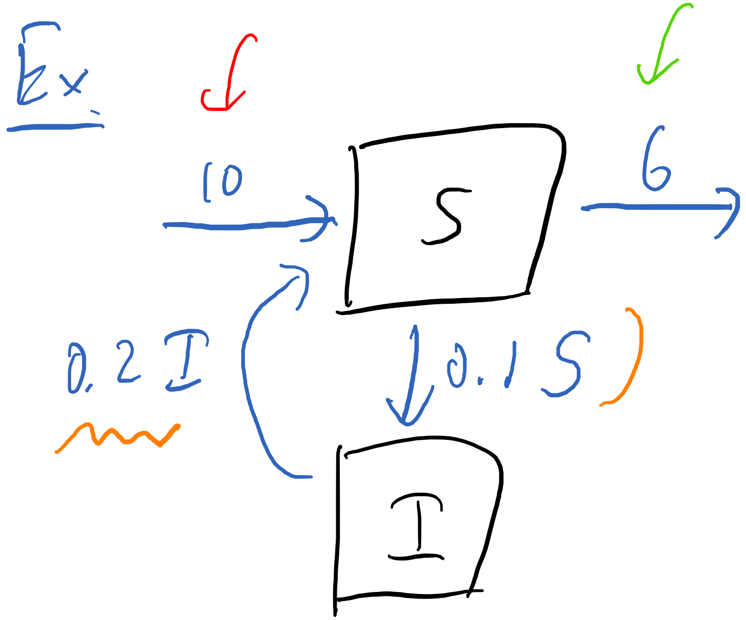
Compartmental diagram: $S' = 10 - \frac{S}{10}$





Compartmental models

- Boxes represent variables
- Arrows give rates of change.
  - An arrow pointing to a box increases that variable
  - An arrow pointing away decreases that variable



system of two ODEs

$$\begin{cases} \dot{S} = 10 - 6 - 0.1S + 0.2I \\ \dot{I} = 0.1S - 0.2I \end{cases}$$

Application: SIR epidemic model

- Consider a simple epidemic model with three classes of people:
  - Susceptible individuals $S(t)$
  - Infected individuals $I(t)$
  - Removed individuals $R(t)$
- Assume that:
  - Infection rate is proportional (β) to the number of infected individuals multiplied by the proportion of susceptible individuals in the population $N = S + I + R$.
  - Recovery rate is proportional (γ) to the number of infected individuals
  - Recovered individuals are immune

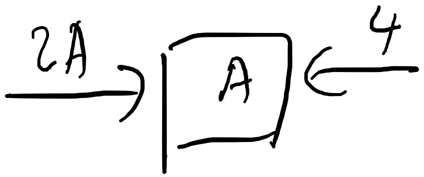
$$\boxed{S} \xrightarrow{\beta I \cdot \frac{S}{N}} \boxed{I} \xrightarrow{\gamma I} \boxed{R}$$

$$\begin{cases} \dot{S} = -\dfrac{\beta S I}{N} \\ \dot{I} = \dfrac{\beta S I}{N} - \gamma I \\ \dot{R} = \gamma I \end{cases}$$

Try it out: $A' + 2A = 4$

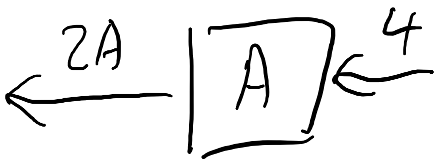
$A' = -2A + 4$

- Does the following compartmentalized model correspond to the equation above?

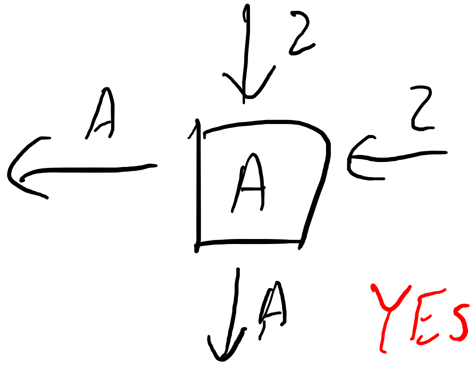


NO

$$\begin{cases} A' = -2A + 4 \\ B' = -4 \end{cases}$$



YES



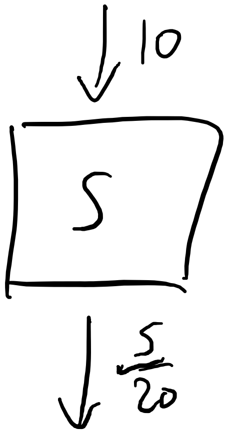
YES

A: Yes
B: No
C: Maybe
D: ???
E: None of the above

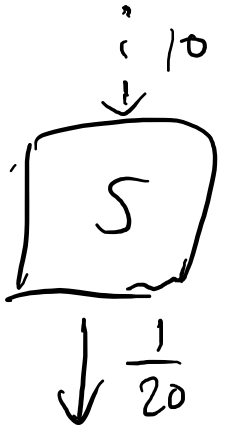
Notational aside

- Bittinger, et al textbook introduces two different types of arrows
  - Relative rates: solid arrows are proportional to the variables
  - Constant/time-dependent rates: dotted arrows are either constant or depend only on time

Ex.



v.s.



textbook

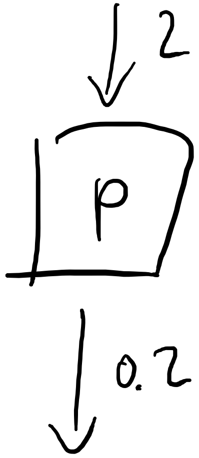
(implicit S)

my notation

- MATA35 will use the left-hand notation of using solid lines everywhere unless explicitly specified otherwise.

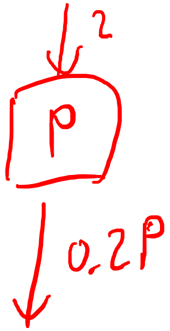
Try it out

- Find the general solution for the following one-compartment model:



$\dot{P} = 2 - 0.2 = 1.8$

$P = 1.8t + C$



$\dot{P} = 2 - 0.2P$

$\dot{P} + 0.2P = 2$

$I(t) = e^{0.2t}$

$Pe^{0.2t} = 10e^{0.2t} + C$

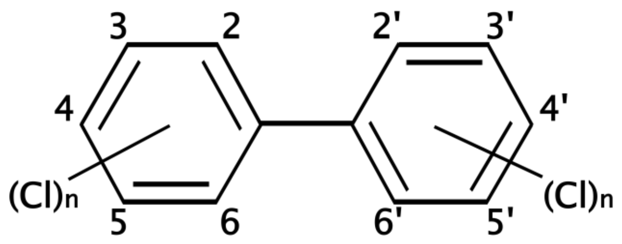
$P = 10 + Ce^{-0.2t}$

A: $P(t) = 1.8t + C$
B: $P(t) = 2.2t + C$
C: $P(t) = 10 + Ce^{-0.2t}$
D: ???
E: None of the above

Lake pollution (Bittinger, pg 557, ex. 5)

unpolluted

- A mussel is placed into lake water polluted by polychlorinated biphenyls (PCBs).
- Let $Q(t)$ be the concentration of PCB in the mussel in micrograms per gram of tissue.
- The mussel absorbs PCBs at 12 micrograms per gram of tissue each day.
- The mussel eliminates PCBs at a rate of $0.18Q$ micrograms per gram of tissue each day.



$\downarrow 12$

$\boxed{Q}$

$\downarrow 0.18Q$

$\dot{Q} = 12 - 0.18Q$

$\dot{Q} + 0.18Q = 12$

$I(t) = e^{0.18t}$

$Qe^{0.18t} = \int 12e^{0.18t}\,dt = \frac{200}{3}e^{0.18t} + C$

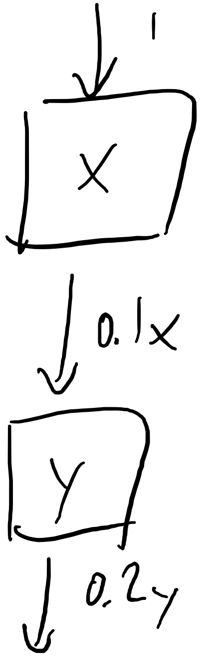
$Q(t) = \frac{200}{3} + Ce^{-0.18t}$

$Q(0) = 0$

$C = -\frac{200}{3}$

$Q(t) = \frac{200}{3} - \frac{200}{3}e^{-0.18t}$

Simple multi-compartmental models



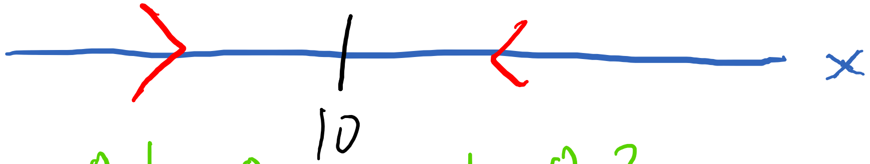
$$\begin{cases} \dot{x} = 1 - 0.1x \\ \dot{y} = 0.1x - 0.2y \end{cases}$$

What are $x(\infty)$ and $y(\infty)$?

$0 = \dot{x} = 1 - 0.1x$

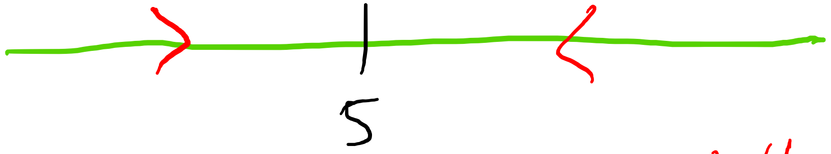
$\Rightarrow x = 10$ is an equilibrium

asymp. stable



At $t = \infty$, $\dot{y} = 0.1x - 0.2y = 1 - 0.2y$

$\Rightarrow y = 5$ is equilibrium



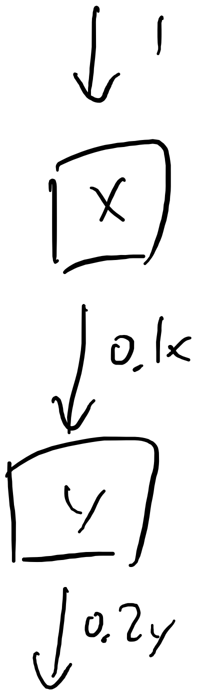
asymp. stable

At time infinity,

$x = 10$

$y = 5$

Solving explicitly



$$\begin{cases} \dot{x} = 1 - 0.1x \\ \dot{y} = 0.1x - 0.2y \end{cases}$$

$$\dot{x} + 0.1x = 1$$

$$I_1(t) = e^{0.1t}$$

$$x e^{0.1t} = \int e^{0.1t}\, dt$$

$$x e^{0.1t} = 10 e^{0.1t} + C_1$$

$$x = 10 + C_1 e^{-0.1t}$$

$$x(t) = 10 + C_1 e^{-0.1t}$$

$$\dot{y} = 0.1\underline{\left(10 + C_1 e^{-0.1t}\right)} - 0.2y$$

$$\dot{y} + 0.2y = 1 + 0.1 e^{-0.1t}$$

$$I_2(t) = e^{0.2t}$$

$$e^{0.2t} \cdot e^{-0.1t} = e^{0.1t}$$

$$y e^{0.2t} = \int \left[e^{0.2t} + 0.1 C_1 e^{0.1t} \right] dt$$

$$y e^{0.2t} = 5 e^{0.2t} + C_1 e^{0.1t} + C_2$$

$$y(t) = 5 + C_1 e^{-0.1t} + C_2 e^{-0.2t}$$

Try it out

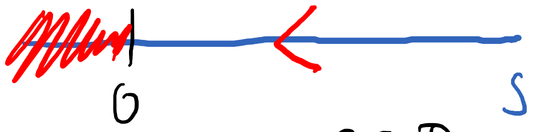
$N = S + I + R$

- What is the behavior of the basic SIR epidemic model with no reinfection at time infinity, if $S(0) > 0, I(0) > 0$, and $R(0) > 0$?

$$S \xrightarrow{\frac{\beta S I}{N}} I \xrightarrow{\gamma I} R$$

$R = N - S - I$

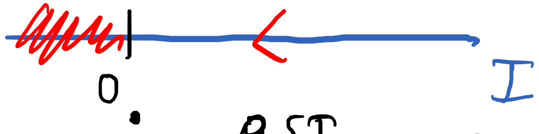
$\lim_{t \to \infty} R(t) = N - 0 - 0 = N$



$\dot{S} = \frac{-\beta S I}{N}$

$S = 0$ is equilibrium

$\lim_{t \to \infty} S(t) = 0$



$\dot{I} = \underbrace{\frac{\beta S I}{N}}_{0 \text{ at } \infty} - \gamma I$

$\lim_{t \to \infty} I(t) = 0$

A: Everyone is susceptible
B: Everyone is infected
C: Everyone is removed
D: Some mix of the above
E: None of the above

Concluding remarks

- Compartmental models are a graphical way of representing the rate of change of variables.
- Linear one-compartment models we can solve using the techniques for linear first-order ODEs
  - Explicitly solving using integrating factors
  - Stability analysis using phase lines
- Multi-compartment models that can be broken down into a series of one-compartment models are also solvable for the same reason.
- More complicated multi-compartment models will require knowing how to solve systems of ODEs. (Future lecture)