Compartmental Models
Lecture 8a – 2021-07-09

MAT A35 – Summer 2021 – UTSC

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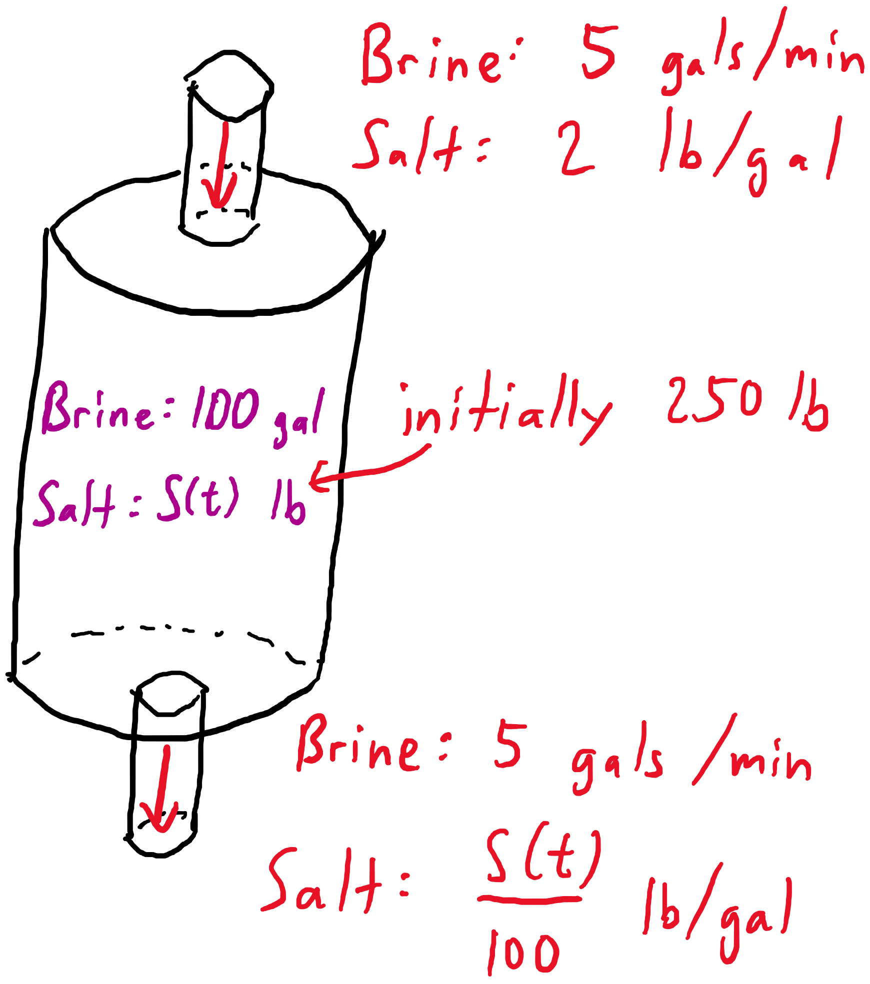
Mixing problem (Bittinger, pg 559, ex. 7)

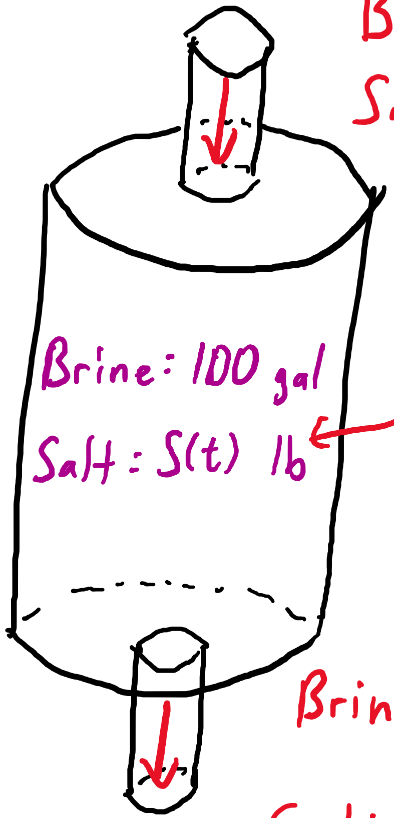
- A tank contains 100 gallons of brine whose concentration is 2.5 lb of salt / gallon.
- Brine containing 2 lb of salt / gallon runs into the tank at a rate of 5 gallons / min
- The brine in the tank runs out at the same rate of 5 gallons / min.
- How does the amount of salt $S(t)$ in the tank change over time?



Rate of change in amount of salt

Brine: 5 gals/min

Salt: 2 lb/gal



initially 250 lb

Brine: 5 gals/min

Salt: $\frac{S(t)}{100}$ lb/gal

Solving for $\dot{S} = 10 - \frac{S}{20}$

Initial value problem

- $S = 200 + Ce^{-\frac{t}{20}}$ and $S(0) = 250$

Amount of salt in tank on long time-scales?

A: 50 lb

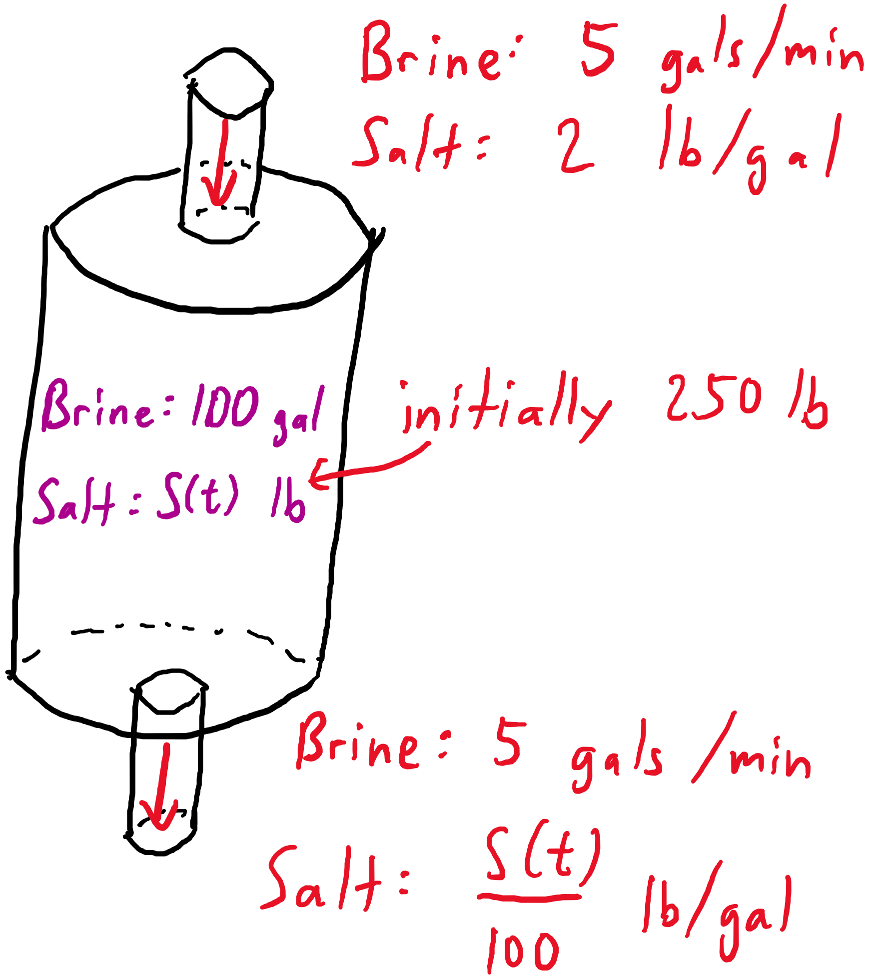
B: 50 lb / min

C: 200 lb

D: 200 lb / min

E: None of the above

Compartmental diagram: $S' = 10 - \frac{S}{10}$



Compartmental models

- Boxes represent variables
- Arrows give rates of change.
    - An arrow pointing to a box increases that variable
    - An arrow pointing away decreases that variable

Application: SIR epidemic model

- Consider a simple epidemic model with three classes of people:
    - Susceptible individuals $S(t)$
    - Infected individuals $I(t)$
    - Removed individuals $R(t)$
- Assume that:
    - Infection rate is proportional to the number of infected individuals multiplied by the proportion of susceptible individuals in the population $N = S + I + R$.
    - Recovery rate is proportional to the number of infected individuals
    - Recovered individuals are immune

Try it out: $A' + 2A = 4$

- Does the following compartmentalized model correspond to the equation above?

A: Yes
B: No
C: Maybe
D: ???
E: None of the above

Notational aside

- Bittinger, et al textbook introduces two different types of arrows
  - Relative rates: solid arrows are proportional to the variables
  - Constant/time-dependent rates: dotted arrows are either constant or depend only on time

- MATA35 will use the left-hand notation of using solid lines everywhere unless explicitly specified otherwise.

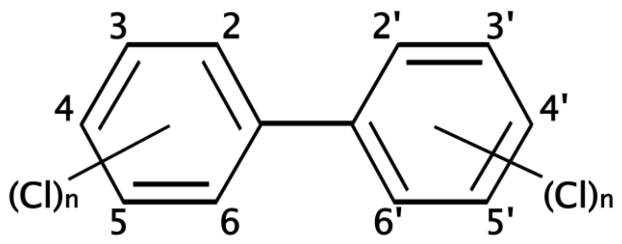
Try it out

- Find the general solution for the following one-compartment model:

A: $P(t) = 1.8t + C$
B: $P(t) = 2.2t + C$
C: $P(t) = 10 + Ce^{-0.2t}$
D: ???
E: None of the above

Lake pollution (Bittinger, pg 557, ex. 5)

- A mussel is placed into lake water polluted by polychlorinated biphenyls (PCBs).
- Let $Q(t)$ be the concentration of PCB in the mussel in micrograms per gram of tissue.
- The mussel absorbs PCBs at 12 micrograms per gram of tissue each day.
- The mussel eliminates PCBs at a rate of $0.18Q$ micrograms per gram of tissue each day.

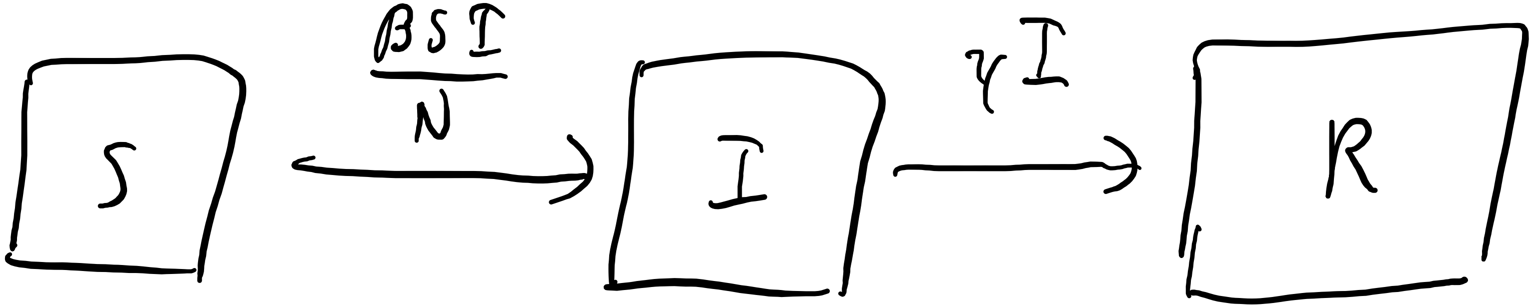


Simple multi-compartmental models

Solving explicitly

Try it out

- What is the behavior of the basic SIR epidemic model with no reinfection at time infinity, if $S(0) > 0, I(0) > 0$, and $R(0) > 0$?



A: Everyone is susceptible
B: Everyone is infected
C: Everyone is removed
D: Some mix of the above
E: None of the above

Concluding remarks

- Compartmental models are a graphical way of representing the rate of change of variables.
- Linear one-compartment models we can solve using the techniques for linear first-order ODEs
  - Explicitly solving using integrating factors
  - Stability analysis using phase lines
- Multi-compartment models that can be broken down into a series of one-compartment models are also solvable for the same reason.
- More complicated multi-compartment models will require knowing how to solve systems of ODEs. (Future lecture)