Numerical solutions: Euler's Method and Runge-Kutta Lecture 8c: 2021-07-14

MAT A35 – Summer 2021 – UTSC

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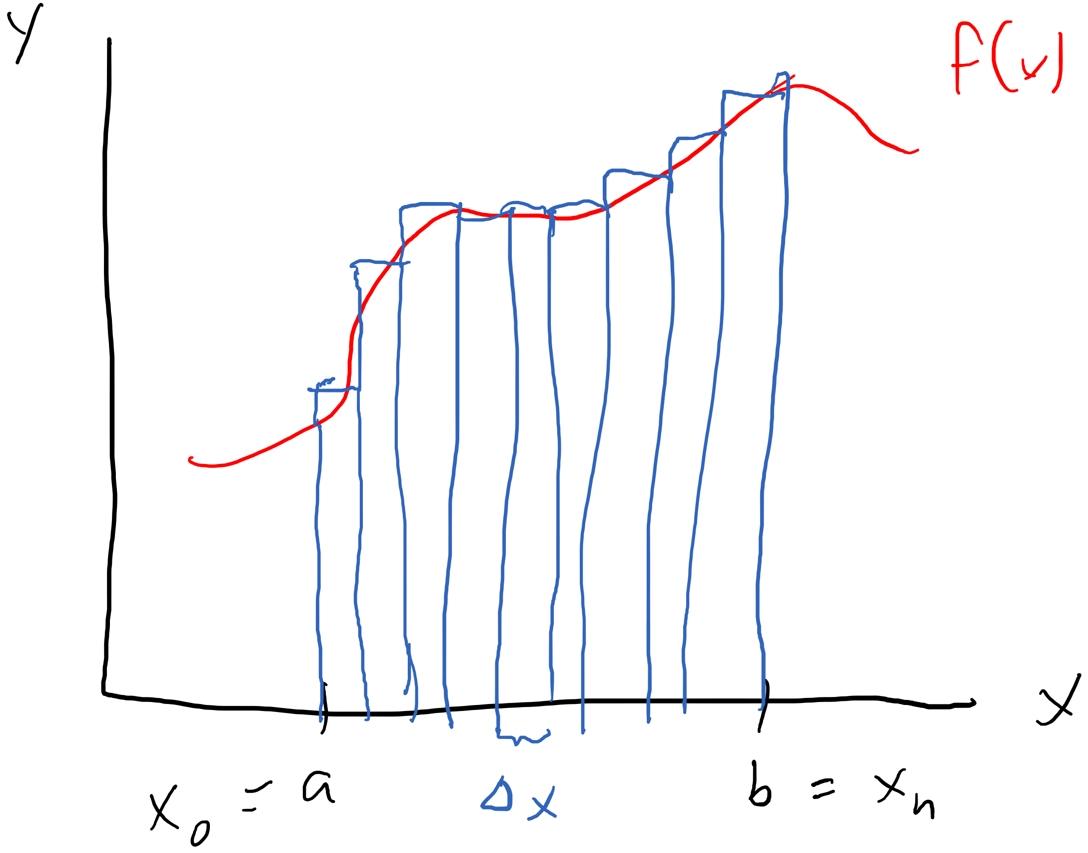
Recall: Riemann Sums

- For any integral problem, we can approximate it with lots of little rectangles. The approximation gets better the more rectangles we have.



$$\int_a^b f(x)\,dx$$

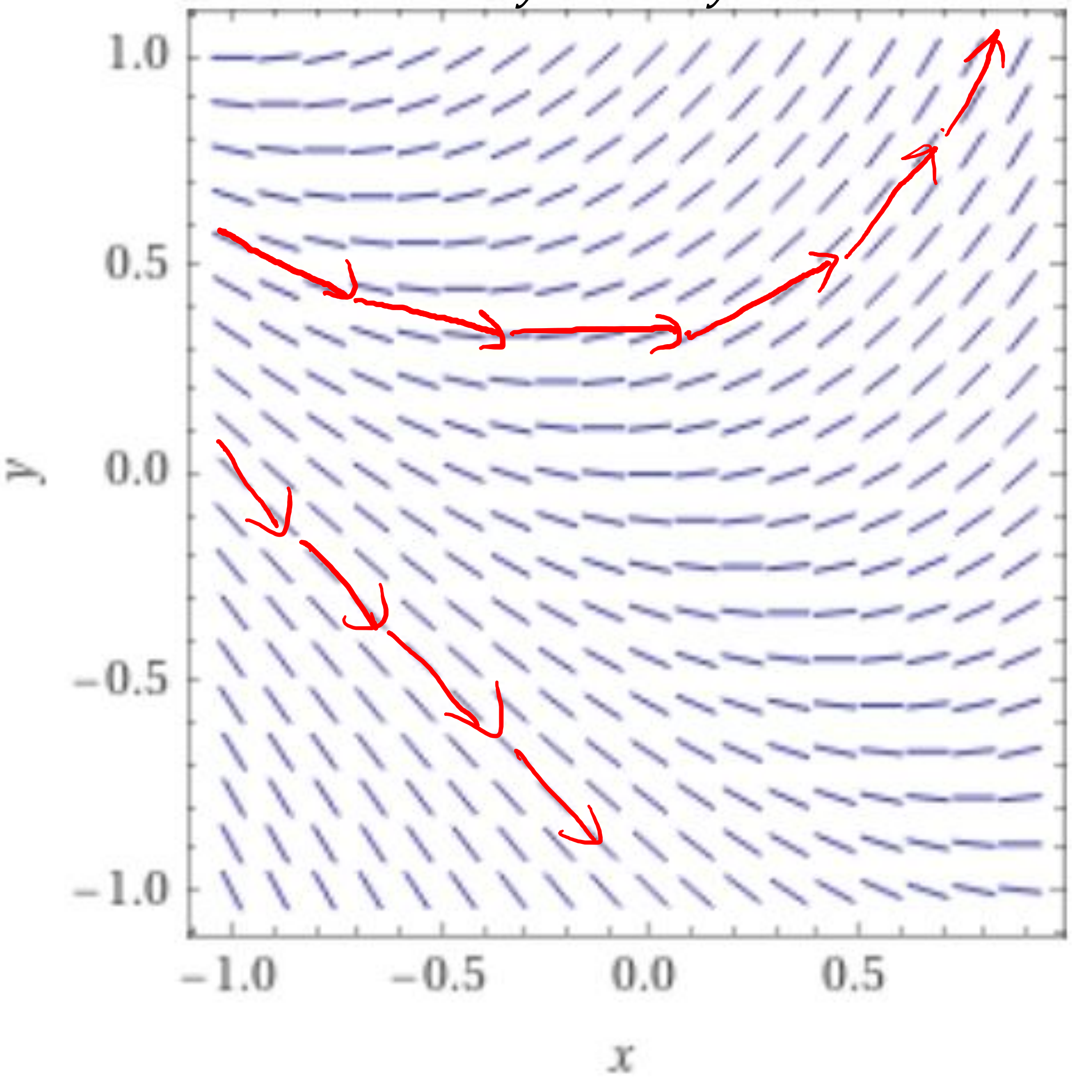
$\approx$ signed area in rectangles

$$= \sum_{i=1}^{n} \Delta x \, f(x_i)$$

Recall: direction fields

- Direction fields tell you what direction a solution to the ODE goes.
- We can approximate a solution to the ODE by starting somewhere and following the direction field.

$y' = x + y$



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Euler's Method

- Suppose we have an IVP
  $y' = f(x, y), \qquad y(x_0) = y_0$
- Choose a *step-size* Δx.
- Then $x_{i+1} = x_i + \Delta x$
- Let $y_{i+1} = y_i + f(x_n, y_n)\Delta x$.
- Then $y_n \approx y(x_n)$.

$$y_{i+1} - y_i = f(x_n, y_n)\Delta x$$

$$\Delta y_i = f(x_n, y_n)\Delta x$$

$$\approx \; dy = f(x,y)\,dx$$

$f(x,y) = x+y$

$y' = x + y \qquad y(-1) = 0.5$

$\Delta x = 0.5$

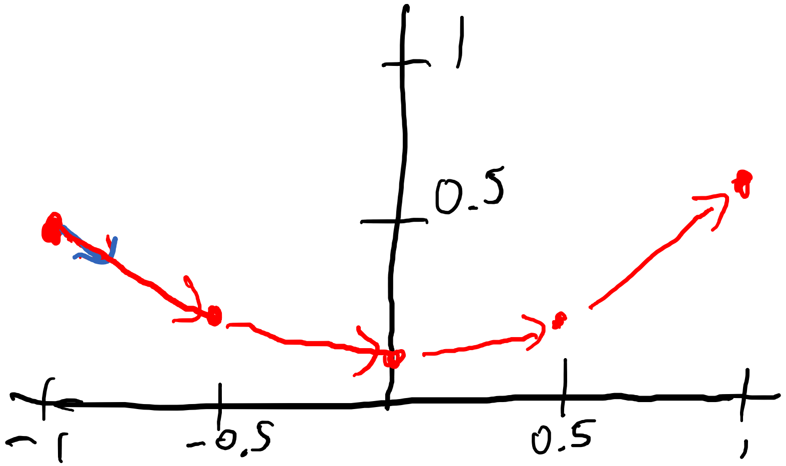
$x_0 = -1 \qquad y_0 = 0.5 \qquad f = -0.5$

$x_1 = -0.5 \qquad y_1 = 0.5 + (-0.5)(0.5) = 0.25 \qquad f = -0.25$

$x_2 = 0 \qquad y_2 = 0.25 + (-0.25)(0.5) = 0.125 \qquad f = 0.125$

$x_3 = 0.5 \qquad y_3 = 0.125 + (0.125)(0.5) = 0.1875 \qquad f = 0.6875$

$x_4 = 1 \qquad y_4 = 0.1875 + (0.6875)(0.5) \approx 0.53$



Try it out

$y(x) =$

$f(x,y) = y$

$y(1) = e \approx 2.718$

- Consider $y' = y$, where $y(0) = 1$. Estimate $y(1)$ using Euler's method with the following step sizes

- $\Delta x = 1$

$$\begin{cases} x_0 = 0 & y_0 = 1 & f = 1 \\ x_1 = 1 & y_1 = 1 + 1.1 = 2 \end{cases}$$

- $\Delta x = \frac{1}{2}$

$$\begin{cases} x_0 = 0 & y_0 = 1 & f(0,1) = 1 \\ x_1 = \frac{1}{2} & y_1 = 1 + 1.0.5 = 1.5 \\ x_2 = 1 & y_2 = 1.5 + 1.5 \cdot 0.5 = 2.25 \end{cases}$$

$y_1 \approx y(0.5)$

- $\Delta x = \frac{1}{3}$

$x_0 = 0 \quad y(0) \approx y_0 = 1 \quad f = 1$

$x_1 = \frac{1}{3} \quad y(\frac{1}{3}) \approx y_1 = \frac{4}{3} \approx 1.\overline{3}$

$x_2 = \frac{2}{3} \quad y(\frac{2}{3}) \approx y_2 = (\frac{4}{3})^2 \approx 1.\overline{77}$

$x_3 = 1 \quad y(1) \approx y_3 = (\frac{4}{3})^3 \approx \overline{2.370}$

Actual solution

$y' = y$

$\frac{dy}{dx} = y$

$\frac{dy}{y} = dx$

$\ln|y| = x + C$

$y = Ce^x$

$y = e^x$

$\Rightarrow |y| = e^{x+C}$

$|y| = e^x \cdot e^C$

$|y| = C e^x$

$y = Ce^x$

$y(0) = 1$

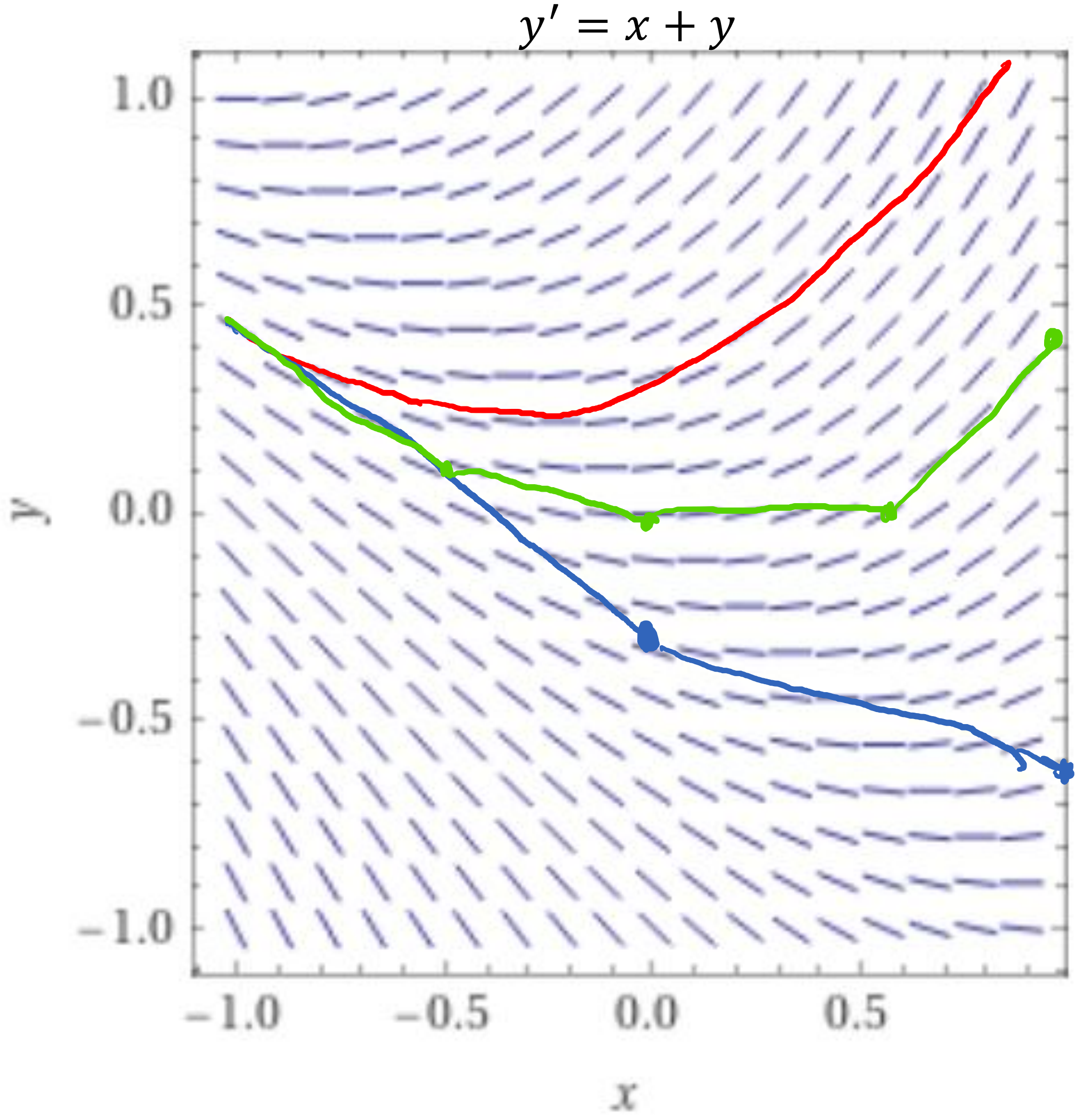
$1 = Ce^0$

$\Rightarrow C = 1$

A: 2.718
B: 2.370
C: 2.250
D: 2.000
E: None of the above

Errors in Euler's method approximations

- We only use the slope at starting point of the integral, and the errors can accumulate.
- The smaller the step size, the more accurate the approximation, but also requires more computation time.



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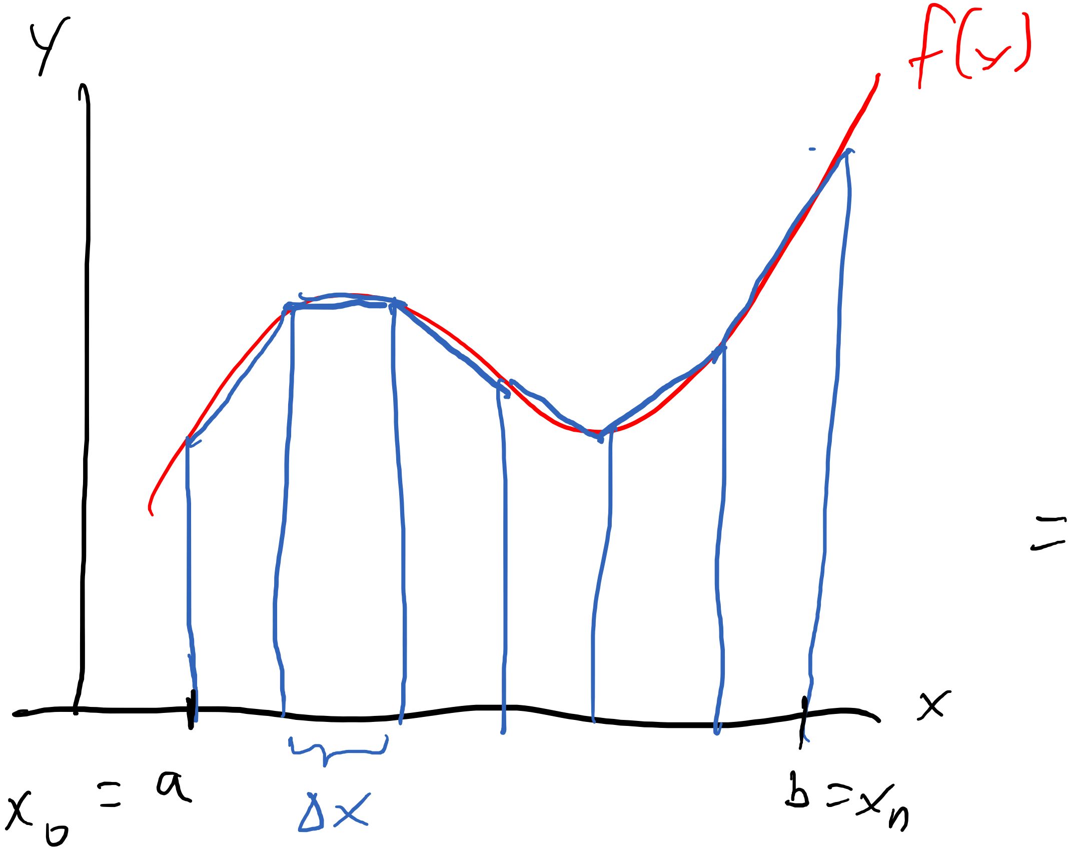
Recall: Trapezoid rule

- We can reduce the error of an integral by using both endpoints of an interval.



$$\int_a^b f(x)\,dx$$

$\approx$ Area in

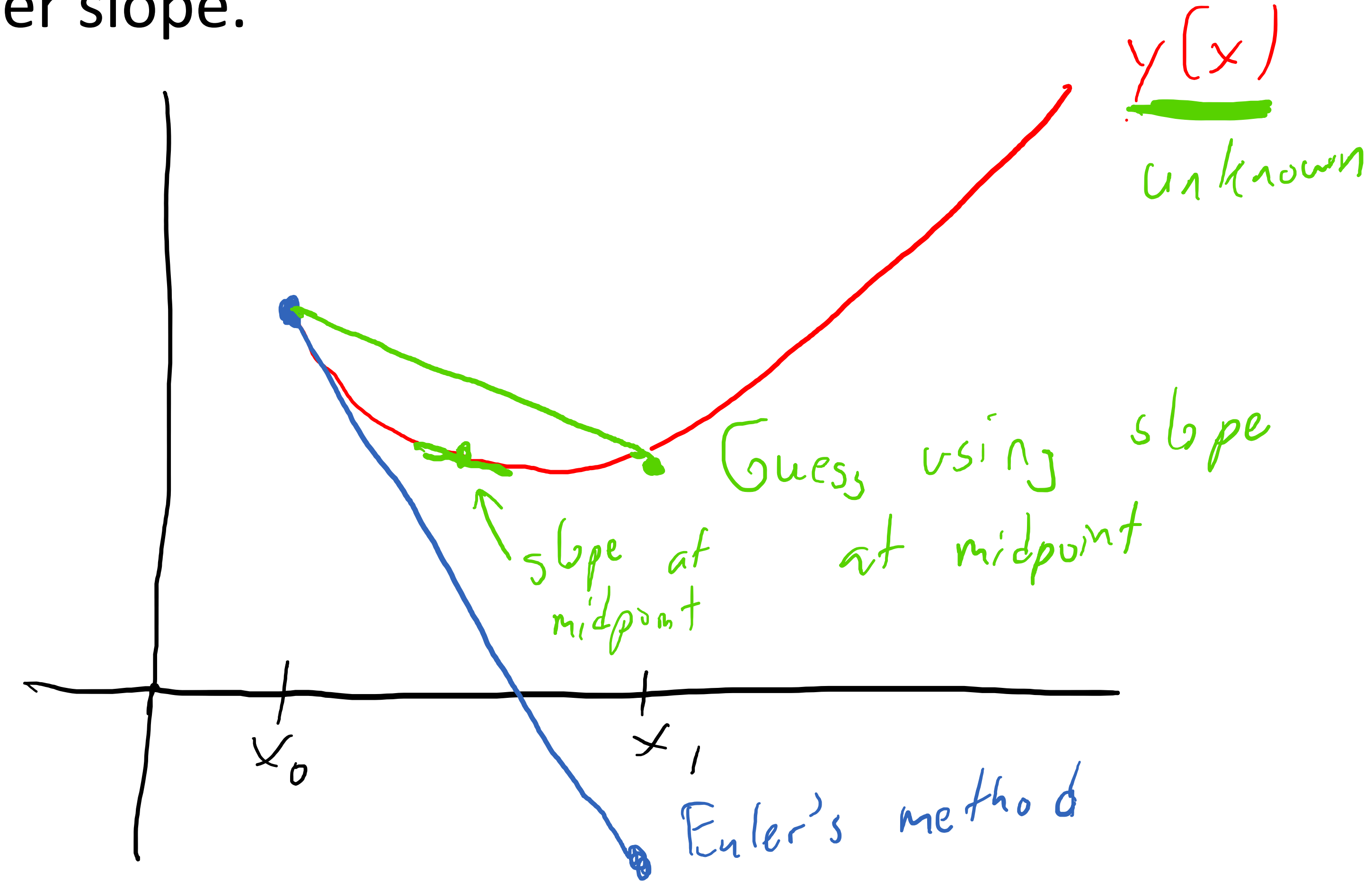
$\approx$ trapezoids

$$= \sum_{i=1}^{n} \frac{1}{2} \Delta x \left(f(x_i) + f(x_{i-1}) \right)$$

Runge-Kutta Family of Methods

- Euler's method is considered 1st-order Runge-Kutta
- Higher-order Runge-Kutta methods use multiple points to derive a better slope.

Intuition



Problem with intuition

- What’s the biggest problem with the intuition on the previous slide?
  - A: We don’t know where the midpoint is (in terms of (x, y) coordinates).
  - B: We know where the midpoint is, but cannot compute the slope there.
  - C: We know where the midpoint is, but its slope is not always a good estimate of the true slope. ← secondary problem
  - D: Computing the midpoint takes a lot of computation.
  - E: None of the above

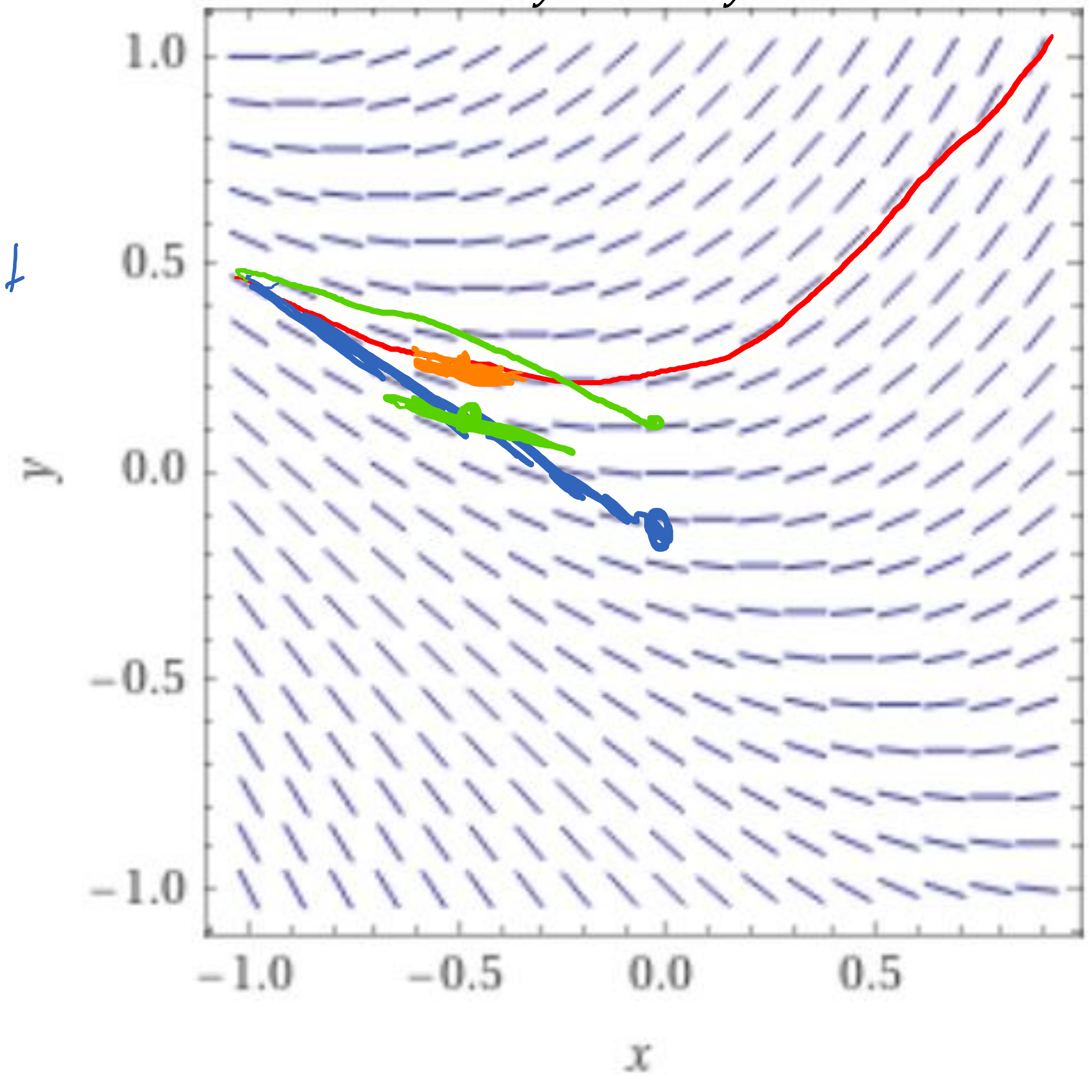
Runge-Kutta – naïve 2nd order midpoint

Intuition

slope at start

slope at guessed midpoint

$y' = x + y$



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Runge-Kutta – naïve 2nd order midpoint

- Suppose we have an IVP

$$y' = f(x, y), \qquad y(x_0) = y_0$$

- Choose a *step-size* Δx. Then $x_{i+1} = x_i + \Delta x$.
- Let $k_1 = f(x_i, y_i)$. slope at starting pt
- Let $k_2 = f\left(\underbrace{x_i + \frac{\Delta x}{2}, y_i + \frac{k_1 \Delta x}{2}}_{\text{guessed midpt based on } k_1}\right)$ slope at guessed midpt
- Let $\underbrace{y_{i+1} = y_i + k_2 \Delta x}_{\text{using slope at guessed midpoint}}$.

Classic Runge-Kutta – 4th order

- Suppose we have an IVP

$$y' = f(x,y), \qquad y(x_0) = y_0$$

- Choose a *step-size* Δx. Then $x_{i+1} = x_i + \Delta x$.
- Let $k_1 = f(x_i, y_i)$. — slope at start
- Let $k_2 = f\left(x_i + \frac{\Delta x}{2}, y_i + \frac{k_1 \Delta x}{2}\right)$ — slope at guessed midpt based on k_1
- Let $k_3 = f\left(x_i + \frac{\Delta x}{2}, y_i + \frac{k_2 \Delta x}{2}\right)$ — slope at guessed mid pt based on k_2
- Let $k_4 = f(x_i + \Delta x, y_i + \mathrm{k}_3 \Delta x)$ — slope at guessed endpt based on k_3

- Let $y_{i+1} = y_i + \underbrace{\frac{1}{6}\Delta x(k_1 + 2\mathrm{k}_2 + 2\mathrm{k}_3 + \mathrm{k}_4)}_{\text{use weighted average of all 4 slopes}}$

Concluding remarks

- Like integrals, solving ODEs explicitly is often hard, and sometimes we don’t have closed-form solutions.
- Like integrals, solving ODEs numerically is actually much easier, since we can approximate by taking lots of tiny Δx steps.
- Euler’s method is similar to Riemann rectangular sums.
- Runge-Kutta (2nd order) is similar to Trapezoid rule.
- Runge-Kutta (classic, 4th order) is similar to Simpson’s rule of thirds.
- In practice, we often solve complicated ODEs using these and other approximations.