Numerical solutions: Euler’s Method and Runge-Kutta
Lecture 8c: 2021-07-14

MAT A35 – Summer 2021 – UTSC

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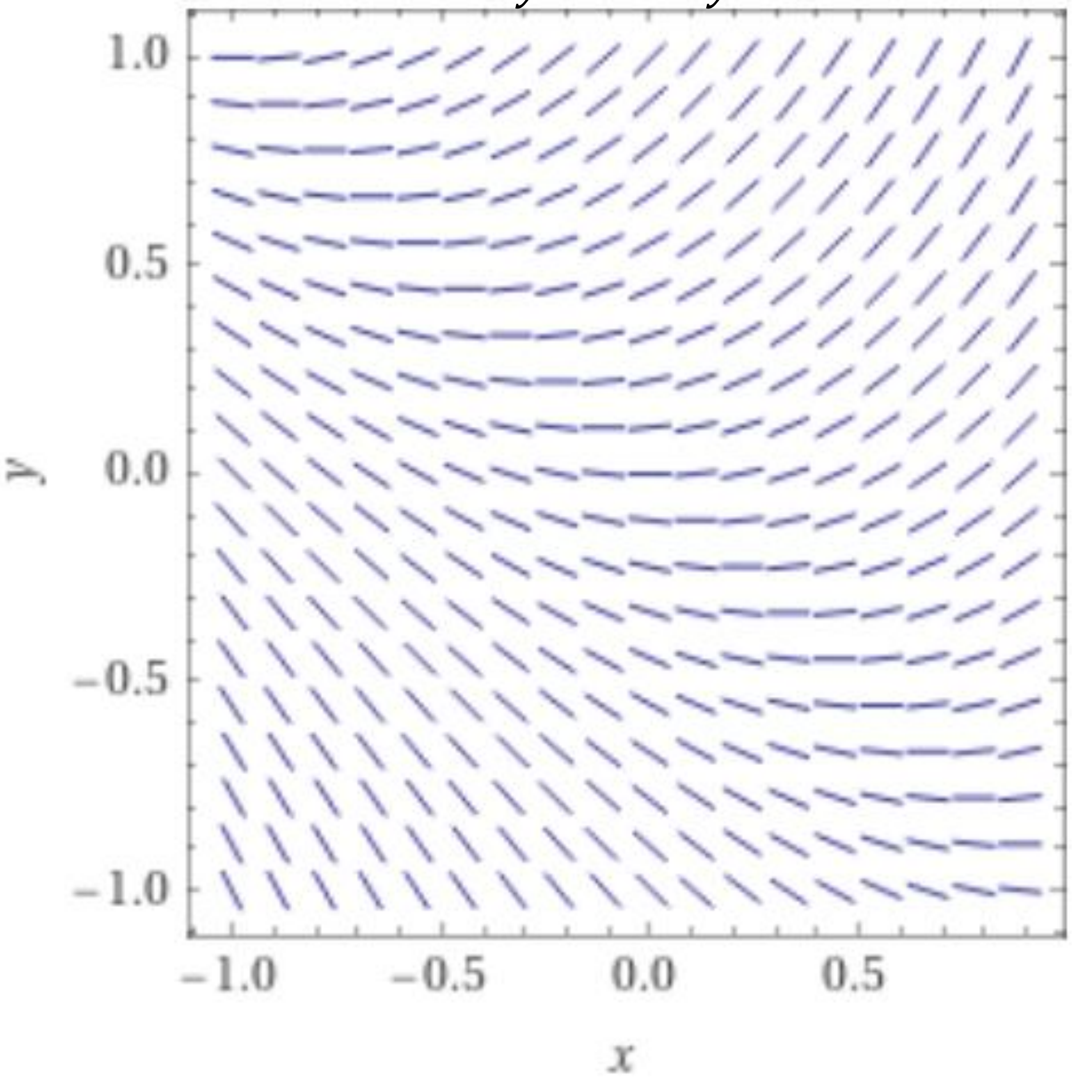
Recall: Riemann Sums

- For any integral problem, we can approximate it with lots of little rectangles. The approximation gets better the more rectangles we have.

Recall: direction fields

- Direction fields tell you what direction a solution to the ODE goes.
- We can approximate a solution to the ODE by starting somewhere and following the direction field.

$y' = x + y$



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Euler's Method

- Suppose we have an IVP
  $y' = f(x, y), \qquad y(x_0) = y_0$
- Choose a *step-size* Δx.
- Then $x_{i+1} = x_i + \Delta x$
- Let $y_{i+1} = y_i + f(x_n, y_n)\Delta x$.
- Then $y_n \approx y(x_n)$.

Try it out

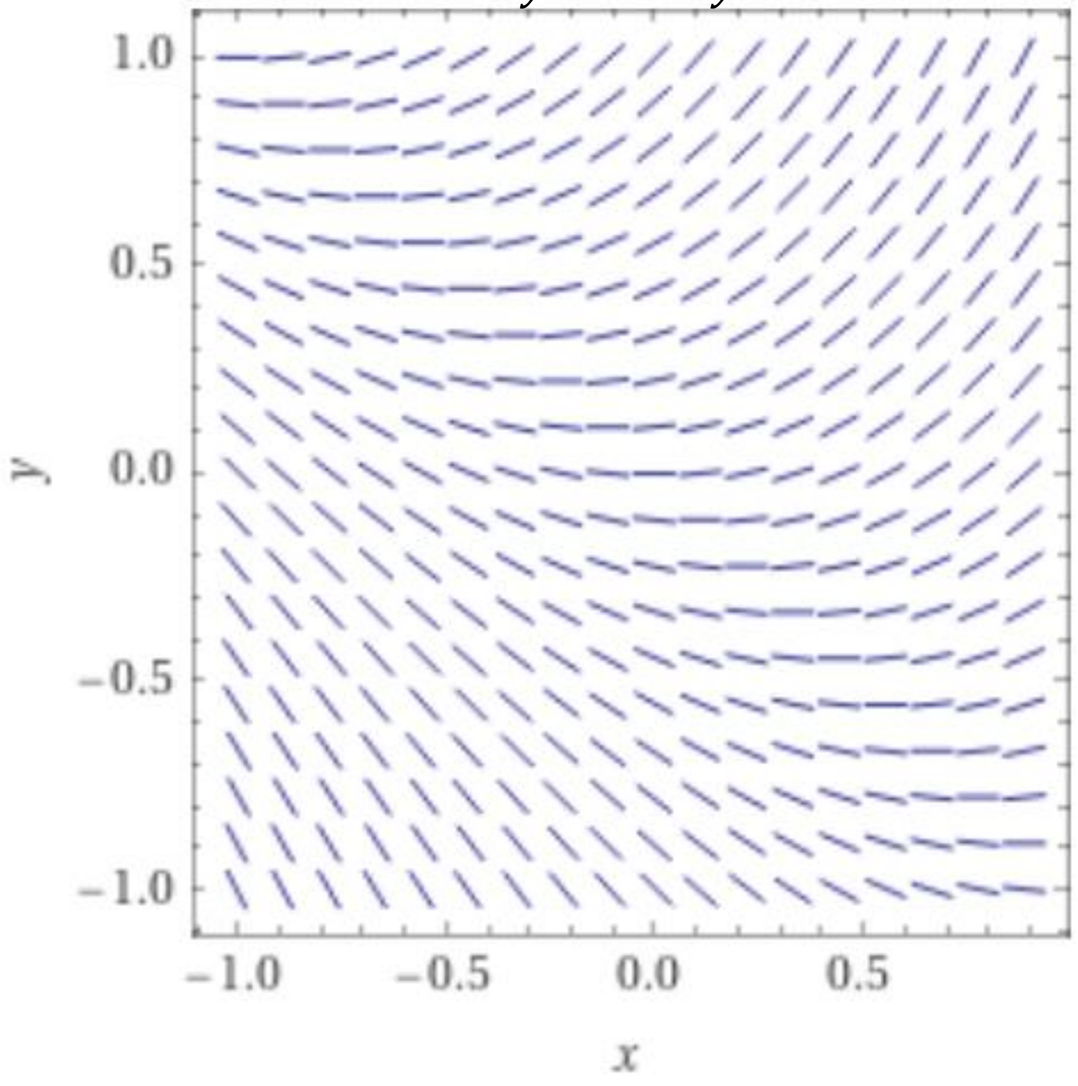
- Consider $y' = y$, where $y(0) = 1$. Estimate $y(1)$ using Euler's method with the following step sizes
- $\Delta x = 1$
- $\Delta x = \frac{1}{2}$
- $\Delta x = \frac{1}{3}$

A: 2.718
B: 2.370
C: 2.250
D: 2.000
E: None of the above

Errors in Euler's method approximations

- We only use the slope at starting point of the integral, and the errors can accumulate.
- The smaller the step size, the more accurate the approximation, but also requires more computation time.

$y' = x + y$



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Recall: Trapezoid rule

- We can reduce the error of an integral by using both endpoints of an interval.

Runge-Kutta Family of Methods

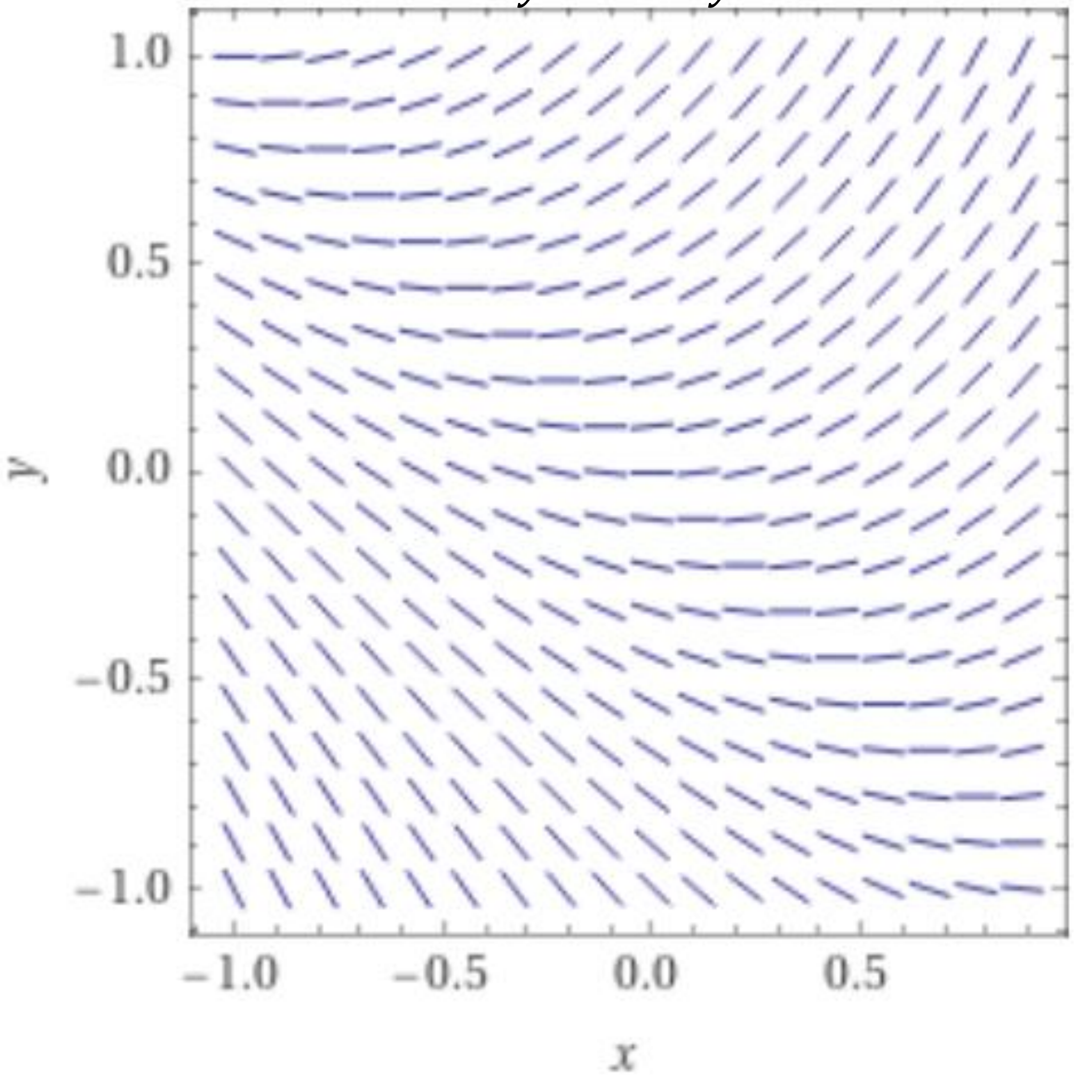
- Euler's method is considered 1st-order Runge-Kutta
- Higher-order Runge-Kutta methods use multiple points to derive a better slope.

Problem with intuition

- What's the biggest problem with the intuition on the previous slide?
  - A: We don't know where the midpoint is (in terms of (x, y) coordinates).
  - B: We know where the midpoint is, but cannot compute the slope there.
  - C: We know where the midpoint is, but its slope is not always a good estimate of the true slope.
  - D: Computing the midpoint takes a lot of computation.
  - E: None of the above

Runge-Kutta – naïve 2nd order midpoint

$y' = x + y$



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Runge-Kutta – naïve 2nd order midpoint

- Suppose we have an IVP

$$y' = f(x, y), \qquad y(x_0) = y_0$$

- Choose a *step-size* Δx. Then $x_{i+1} = x_i + \Delta x$.
- Let $k_1 = f(x_i, y_i)$.
- Let $k_2 = f\left(x_i + \frac{\Delta x}{2}, y_i + \frac{k_1 \Delta x}{2}\right)$
- Let $y_{i+1} = y_i + k_2 \Delta x$.

Classic Runge-Kutta – 4th order

- Suppose we have an IVP
$$y' = f(x, y), \qquad y(x_0) = y_0$$
- Choose a *step-size* Δx. Then $x_{i+1} = x_i + \Delta x$.
- Let $k_1 = f(x_i, y_i)$.
- Let $k_2 = f\left(x_i + \frac{\Delta x}{2}, y_i + \frac{k_1 \Delta x}{2}\right)$
- Let $k_3 = f\left(x_i + \frac{\Delta x}{2}, y_i + \frac{k_2 \Delta x}{2}\right)$
- Let $k_4 = f(x_i + \Delta x, y_i + \mathrm{k}_3 \Delta x)$

- Let $y_{i+1} = y_i + \frac{1}{6}\Delta x(k_1 + 2\mathrm{k}_2 + 2\mathrm{k}_3 + \mathrm{k}_4)$

Concluding remarks

- Like integrals, solving ODEs explicitly is often hard, and sometimes we don't have closed-form solutions.
- Like integrals, solving ODEs numerically is actually much easier, since we can approximate by taking lots of tiny Δx steps.
- Euler's method is similar to Riemann rectangular sums.
- Runge-Kutta (2^{nd} order) is similar to Trapezoid rule.
- Runge-Kutta (classic, 4^{th} order) is similar to Simpson's rule of thirds.
- In practice, we often solve complicated ODEs using these and other approximations.