Constant coefficient homogeneous higher-order linear ODEs
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MAT A35 – Summer 2021 – UTSC

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Recall: linear higher-order ODEs

- Linear ODEs: $a_n(x)y^{(n)} + \cdots a_1(x)y' + a_0(x)y = q(x)$, where $a_i(x)$ and $q(x)$ are all functions of x.

Ex. $y'' + y' + y = 5$ $\cancel{y'' + y' + y^2 = 5}$

$y''' + \sin(x)\,y' + x^2 y = 5x$ $\cancel{y''' + \sin(y)\,y' + x^2 y = 5}$

$y''' + \sin(x)y' + 5x^2 y = 5y$

$\Rightarrow y''' + \sin(x)y' + (5x^2 - 5)y = 0$ ✓

A: Linear
B: Nonlinear
C: Both
D: ???
E: None of the above

(In)homogeneous linear ODEs

- Linear ODEs: $a_n(x)y^{(n)} + \cdots a_1(x)y' + a_0(x)y = q(x)$, where $a_i(x)$ and $q(x)$ are all functions of x.
  - If $q(x) = 0$, then *homogeneous.*
  - Otherwise, it is *inhomogeneous*.
  - Note, if nonlinear, then neither definition applies.

$y'' + y' + y = 5$ ← inhomogeneous → $y'' + y' + y - 5 = 0$

?

$y'' + y' + y = 0$ homogeneous

$y''' + \sin(x) y' + x^2 y = 5y$

$y''' + \sin(x) y' + (x^2 - 5) y = 0$ homogeneous

Constant coefficient linear ODEs

- Linear ODEs: $a_n(x)y^{(n)} + \cdots a_1(x)y' + a_0(x)y = q(x)$, where $a_i(x)$ and $q(x)$ are all functions of x.
  - If $a_i(x) = a_i$ for some constant a_i, then it has constant coefficients
  - Otherwise, is does not have constant coefficients
  - Note, if nonlinear, this terminology does not apply.

$1y'' + 1y' + 2y = 0$ constant coef.

$y'' + \cancel{5}xy' + y = 0$ nonconstant coef.

$y'' + 5y' + y = 5x$ constant coef. / nonautonomous

↳ doesn't matter for constant coef.

Try it out: homogeneity and coefficients?

constant

inhomog.

- $2y' + 9y = \boxed{x^2}$ B
- $2y' - \pi y = 0$ homog. A
- $y'' + \boxed{x}y' + y = 0$ C
- $y'' + \boxed{e^x}y' = \boxed{3}$ D
- $y'' - 2y + y^2 = 5$ E
- $\ddot{x} + 4\dot{x} = -4x$
- $(\sin x)y'' + e^x y' + y = 0$ C
- $xy'' + y = x^2$ D
- $y'' + 4y + 4 = 0$

A: Homogeneous, constant coefficients
B: Inhomogeneous, constant coefficients
C: Homogeneous, nonconstant coefficients
D: Inhomogeneous, nonconstant coefficients
E: None of the above

$y'' + y = -xy'$

nonlinear

$\ddot{x} + 4\dot{x} + 4x = 0$ A

$y'' + 4y = -4$ B

$\frac{dy}{dx}$ dep. ind.

$\frac{dx}{dt}$ dep. ind.

Scaling of sols to homogeneous eq

$\checkmark\ y_1(x)$

- Let y_1 be a sol. to the homogeneous linear ODE

$$a_n(x)y^{(n)} + \cdots a_1(x)y' + a_0(x)y = 0$$

- Then c_1y_1 is a solution to the same ODE, where c_1 is a constant.

proof.

$$a_n \cdot \frac{d^n}{dx^n}(c_1y_1) + \cdots + a_1 \frac{d}{dx}(c_1y_1) + a_0c_1y_1 = 0$$

$$= c_1 \underbrace{\left[a_n \cdot \frac{d^n}{dx^n}(y_1) + \cdots + a_1 \frac{d}{dx}(y_1) + a_0y_1\right]}_{0} = 0$$

$\square$

Ex.

$y'' + 3y' + 2y = 0$ → Check $y_1 = e^{-x} + (-3e^{-x}) + 2e^{-x} = 0\ \checkmark$

$y_1 = e^{-x}$

Check $5y_1 = 5e^{-x} - 15e^{-x} + 10e^{-x} = 0\ \checkmark$

Adding sols to homogeneous equation

- Let $y_1(x)$ and $y_2(x)$ be sol. to the homogeneous linear ODE

$$a_n(x)y^{(n)} + \cdots a_1(x)y' + a_0(x)y = 0$$

- Then $y_1 + y_2$ is a solution to the same ODE.

proof. $a_n \frac{d^n}{dx^n}[y_1+y_2] + \cdots + a_1 \frac{d}{dx}[y_1+y_2] + a_0[y_1+y_2]$

$$= \underbrace{[a_n y_1^{(n)} + \cdots + a_1 y_1' + a_0 y_1]}_{0} + \underbrace{[a_n y_2^{(n)} + \cdots + a_1 y_2' + a_0 y_2]}_{0} = 0 \quad \boxed{/}$$

Ex, $y'' + 3y' + 2y = 0$ $\quad y_1 = e^{-x} \quad y_2 = e^{-2x}$

Check $y_1 + y_2$:

$$[e^{-x} + e^{-2x}]'' + 3[e^{-x} + e^{-2x}]' + 2[e^{-x} + e^{-2x}]$$

$$= e^{-x} + 4e^{-2x} - 3e^{-x} - 6e^{-2x} + 2e^{-x} + 2e^{-2x}$$

$$= 0$$

Main Theorems

- Let $y_1(x), y_2(x), \ldots, y_n(x)$ be solutions to the homogeneous linear ODE
$$a_n(x)y^{(n)} + \cdots a_1(x)y' + a_0(x)y = 0$$
- Principal of Superposition: then $c_1y_1 + c_2y_2 + \cdots + c_ny_n$ is a solution to the same ODE, where c_i are arbitrary constants.
- General solution: If $y_1, \ldots, y_n$ are linearly independent, then *all* solutions to the ODE can be written in the form
$$c_1y_1 + c_2y_2 + \cdots + c_ny_n$$
so we call that the general solution to the ODE.

Recall: independent means $c_1y_1 + \cdots + c_ny_n = 0$ only if $c_1 = 0, c_2 = 0, \ldots, c_n = 0$.

Constant coefficient homogeneous sol

- Consider $a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0$, where a_i are constant.
- We can write a characteristic polynomial
$$p(r) = a_n r^n + \cdots + a_1 r + a_0$$
- If λ is a root of the polynomial (i.e. $p(\lambda) = 0$), then $e^{\lambda x}$ is a solution to the ODE.
- If λ is a root of the polynomial with multiplicity k, then $x^{k-1} e^{\lambda x}$ is a solution to the ODE.
- Note, we will often call λ an eigenvalue of the ODE, for reasons that will become clear later.

Example

$$\left[\frac{d^2}{dx^2}\right]y + \left[\frac{d}{dx}\right]y + 2y \longrightarrow \text{replace } \frac{d}{dx} \text{ with } r.$$

$y'' + 3y' + 2y = 0$

$p(r) = r^2 + 3r + 2$

$p(\lambda) = 0 = \lambda^2 + 3\lambda + 2$

$0 = (\lambda+1)(\lambda+2)$

$\lambda = -1, \ -2$

Thus, e^{-x}, e^{-2x} are sols.

$c_1 e^{-x} + c_2 e^{-2x}$

$y'' + 2y' + y = 0$

$p(r) = r^2 + 2r + 1$

$p(\lambda) = \lambda^2 + 2\lambda + 1 = 0$

$(\lambda+1)(\lambda+1) = 0$

$\lambda = -1$, multiplicity 2

Then e^{-x}, xe^{-x} are sols.

$c_1 e^{-x} + c_2 xe^{-x}$

Intuitive proof idea

$$y'' + 3y' + 2y = 0$$

$$\frac{d^2}{dx^2} y + 3 \cdot \frac{d}{dx} y + 2y = 0$$

$$\left(\frac{d^2}{dx^2} + 3\frac{d}{dx} + 2\right) y = 0$$

$$\left(\frac{d}{dx} + 1\right) \underbrace{\left(\frac{d}{dx} + 2\right) y}_{} = 0$$

or

$$\left(\frac{d}{dx} + 2\right) \underbrace{\left(\frac{d}{dx} + 1\right) y}_{} = 0$$

Lemma: $\left(\frac{d}{dx} + \lambda\right) y = 0$

$$\frac{dy}{dx} + \lambda y = 0$$

$$dy = -\lambda y \, dx$$

$$\int \frac{dy}{y} = \int -\lambda \, dx$$

$$\ln |y| = -\lambda x + C$$

$$|y| = Ce^{-\lambda x}, \quad C \text{ pos}$$

$$y = Ce^{-\lambda x}, \quad \text{any } C$$

- $y_1 = Ce^{-2x}$
- $y_2 = Ce^{-x}$

Try it out

- Which of the following are solutions to

$$y''' - 2y'' - y' + 2y = 0?$$

Char. eq. $\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$

$(\lambda - 2)(\lambda^2 - 1) = 0$

$(\lambda - 2)(\lambda + 1)(\lambda - 1) = 0$

$\lambda = -1, 1, 2$

Solutions: e^{-x}, e^{x}, e^{2x}

$c_1 e^{-x} + c_2 e^{x} + c_3 e^{2x}$

A: e^{-x}
B: e^{2x}
C: $e^{-x} + 5e^{x} - 2e^{2x}$
D: All of the above
E: None of the above

Try it out

- What is the general solution to $y'' + 4y' + 4y = 0$?

$\lambda^2 + 4\lambda + 4 = 0$

$(\lambda+2)^2 = 0$

$\lambda = -2$, multiplicity 2

e^{-2x}, xe^{-2x} are solutions

A: c_1e^{-2x}
B: c_1xe^{-2x}
C: $c_1e^{-2x} + c_2xe^{-2x}$
D: All of the above
E: None of the above

- What is the solution to the IVP given $y'' + 4y' + 4y = 0$, $y(0) = 1, y'(0) = 2$?

$y(x) = c_1 e^{-2x} + c_2 x e^{-2x}$

$y'(x) = -2c_1 e^{-2x} - 2c_2 x e^{-2x} + c_2 e^{-2x}$

$1 = y(0) = c_1 + c_2$

$2 = y'(0) = -2c_1 + c_2$

$c_1 = -\frac{1}{3}$ $c_2 = \frac{4}{3}$

$y(x) = -\frac{1}{3}e^{-2x} + \frac{4}{3}xe^{-2x}$

A: $e^{-2x} + 2xe^{-2x}$
B: $-\frac{1}{3}e^{-2x} + \frac{4}{3}xe^{-2x}$
C: $-\frac{1}{2}e^{-2x} + 2c_2xe^{-2x}$
D: All of the above
E: None of the above