Constant coefficient homogeneous higher-order linear ODEs
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Recall: linear higher-order ODEs

- Linear ODEs: $a_n(x)y^{(n)} + \cdots a_1(x)y' + a_0(x)y = q(x)$, where $a_i(x)$ and $q(x)$ are all functions of x.

A: Linear
B: Nonlinear
C: Both
D: ???
E: None of the above

(In)homogeneous linear ODEs

- Linear ODEs: $a_n(x)y^{(n)} + \cdots a_1(x)y' + a_0(x)y = q(x)$, where $a_i(x)$ and $q(x)$ are all functions of x.
  - If $q(x) = 0$, then *homogeneous.*
  - Otherwise, it is *inhomogeneous*.
  - Note, if nonlinear, then neither definition applies.

Constant coefficient linear ODEs

- Linear ODEs: $a_n(x)y^{(n)} + \cdots a_1(x)y' + a_0(x)y = q(x)$, where $a_i(x)$ and $q(x)$ are all functions of x.
  - If $a_i(x) = a_i$ for some constant a_i, then it has constant coefficients
  - Otherwise, is does not have constant coefficients
  - Note, if nonlinear, this terminology does not apply.

Try it out: homogeneity and coefficients?

- $y' + 9y = x^2$
- $y' - \pi y = 0$
- $y'' + xy' + y = 0$
- $y'' + e^x y' = 3$
- $y'' - 2y + y^2 = 5$
- $\ddot{x} + 4\dot{x} = -4x$
- $(\sin x)y'' + e^x y' + y = 0$
- $xy'' + y = x^2$
- $y'' + 4y + 4 = 0$

A: Homogeneous, constant coefficients
B: Inhomogeneous, constant coefficients
C: Homogeneous, nonconstant coefficients
D: Inhomogeneous, nonconstant coefficients
E: None of the above

Scaling of sols to homogeneous eq

- Let y_1 be a sol. to the homogeneous linear ODE
$$a_n(x)y^{(n)} + \cdots a_1(x)y' + a_0(x)y = 0$$
- Then $c_1 y_1$ is a solution to the same ODE, where c_1 is a constant.

Adding sols to homogeneous equation

- Let $y_1(x)$ and $y_2(x)$ be sol. to the homogeneous linear ODE
$$a_n(x)y^{(n)} + \cdots a_1(x)y' + a_0(x)y = 0$$
- Then $y_1 + y_2$ is a solution to the same ODE.

Main Theorems

- Let $y_1(x), y_2(x), \ldots, y_n(x)$ be solutions to the homogeneous linear ODE
$$a_n(x)y^{(n)} + \cdots a_1(x)y' + a_0(x)y = 0$$
- Principal of Superposition: then $c_1y_1 + c_2y_2 + \cdots + c_ny_n$ is a solution to the same ODE, where c_i are arbitrary constants.
- General solution: If $y_1, \ldots, y_n$ are linearly independent, then *all* solutions to the ODE can be written in the form
$$c_1y_1 + c_2y_2 + \cdots + c_ny_n$$
so we call that the general solution to the ODE.

Constant coefficient homogeneous sol

- Consider $a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0$, where a_i are constant.
- We can write a characteristic polynomial
$$p(r) = a_n r^n + \cdots + a_1 r + a_0$$
- If λ is a root of the polynomial (i.e. $p(\lambda) = 0$), then $e^{\lambda x}$ is a solution to the ODE.
- If λ is a root of the polynomial with multiplicity k, then $x^{k-1} e^{\lambda x}$ is a solution to the ODE.
- Note, we will often call λ an eigenvalue of the ODE, for reasons that will become clear later.

Example

Intuitive proof idea

Try it out

- Which of the following are solutions to

$$y''' - 2y'' - y' + 2y = 0?$$

A: e^{-x}
B: e^{2x}
C: $e^{-x} + 5e^{x} - 2e^{2x}$
D: All of the above
E: None of the above

Try it out

- What is the general solution to $y'' + 4y' + 4y = 0$?

A: $c_1 e^{-2x}$
B: $c_1 x e^{-2x}$
C: $c_1 e^{-2x} + c_2 x e^{-2x}$
D: All of the above
E: None of the above

- What is the solution to the IVP given $y'' + 4y' + 4y = 0$, $y(0) = 1, y'(0) = 2$?

A: $e^{-2x} + 2xe^{-2x}$
B: $-\frac{1}{3}e^{-2x} + \frac{4}{3}xe^{-2x}$
C: $-\frac{1}{2}e^{-2x} + 2c_2 xe^{-2x}$
D: All of the above
E: None of the above