Complex numbers and rotations
Lecture 9b: 2021-07-21

MAT A35 – Summer 2021 – UTSC

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No algebraic closure in real numbers $\mathbb{R}$

- Problem: no real solution to the equation $x^2 + 1 = 0$.

- Problem: no real solution to the equation $x^2 + 2x + 2 = 0$

- *Algebraic closure* of the reals means that every polynomial $P(x)$ has to have a real root $P(z) = 0$, but this is not true.

Imagining up a new number

- Let's start by defining a solution to $x^2 + 1 = 0$:

$$x^2 + 1 = 0$$
$$x^2 = -1$$
$$x = \pm\sqrt{-1}$$

- Let $i = \sqrt{-1}$.

Complex numbers

- Since i is a number, we want to be able to add, subtract, multiply, and divide with it, like with real numbers.

- We call all of these new “numbers” the *complex numbers* $\mathbb{C}$.

Canonical form of complex numbers

- It turns out that every complex number $z \in \mathbb{C}$ can be written simply as $z = a + bi$, where $a, b \in \mathbb{R}$, and $i = \sqrt{-1}$.

Algebraic closure of complex numbers

- $z \in \mathbb{C}$ if $z = a + bi$, where $a, b \in \mathbb{R}$, and $i = \sqrt{-1}$.
- It turns out that $\mathbb{C}$ is *algebraically closed*, which means that any non-constant polynomial has a root in $\mathbb{C}$.

Complex plane

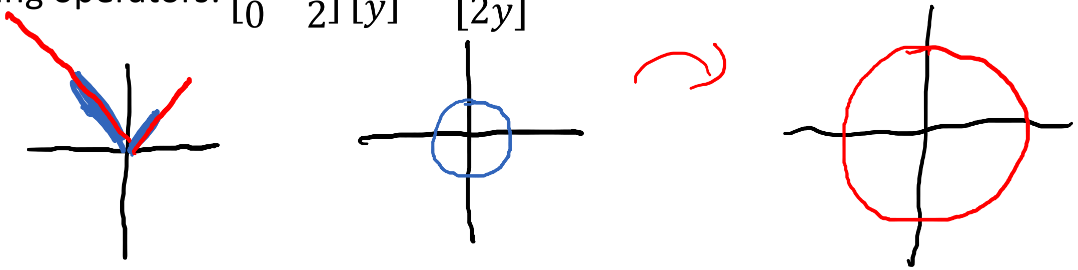
- We can use tools from linear algebra to understand $a + bi$.
- Since $a, b \in \mathbb{R}$, we can think of the point $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$ as a way to represent $a + bi$.

What does this have to do with biology?

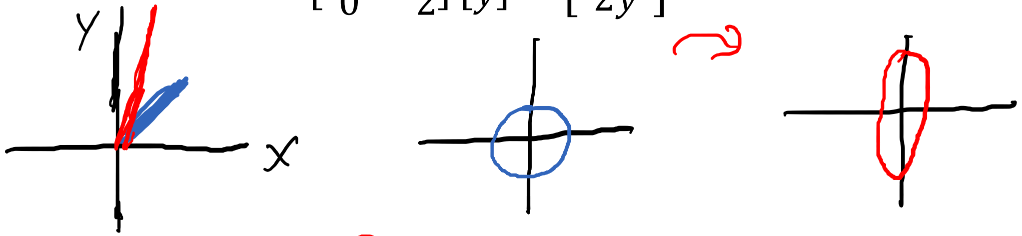
- When talking about population sizes "negative" population sizes were considered meaningless because we can't have negative numbers of animals.
- What does it mean to have "imaginary" numbers of bunnies or birds?

Recall: Matrices are transformations of vectors

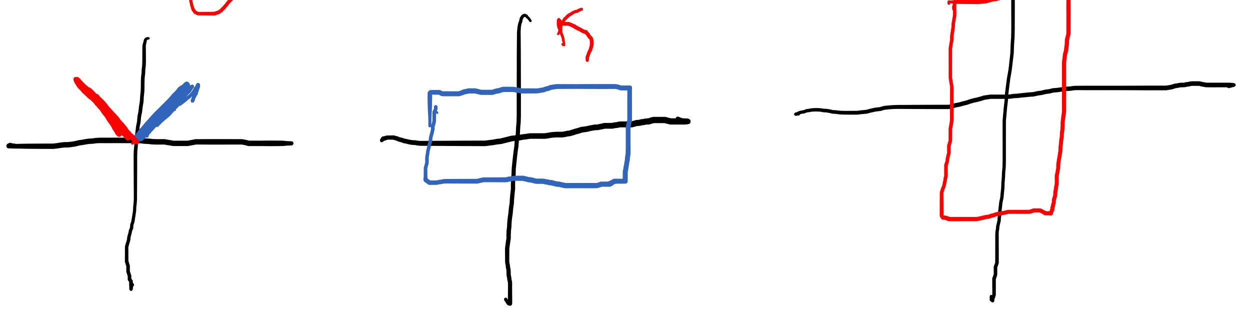
- Scaling operators: $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$



- Stretching/squashing: $\begin{bmatrix} 0.5 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.5x \\ 2y \end{bmatrix}$



- Rotations: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$



Multiplication by i = rotation by 90°

Multiplication by $2i$ = scaling by 2 + rotation by 90°

What about other rotation angles?

- We want something to multiply the basis vector 1 by that leaves you with something of length 1 that has the correct angle.
- But multiplying by 1 is the identity.

Rotation by θ = multiply by $\cos\theta + i\sin\theta$

Multiplication by $\mathrm{z} = a + bi$

- Notice that we can think of all complex multiplications as a rotation and then a scaling.
- The length of the scaling is the *modulus* $|z| = \sqrt{a^2 + b^2}$
- The angle of the rotation is the *argument* $\theta = Arg(z)$, where $\frac{a+bi}{|z|} = \cos\theta + i\sin\theta$

Try it out

- $|-1+i| =$

- $Arg(-1+i) =$

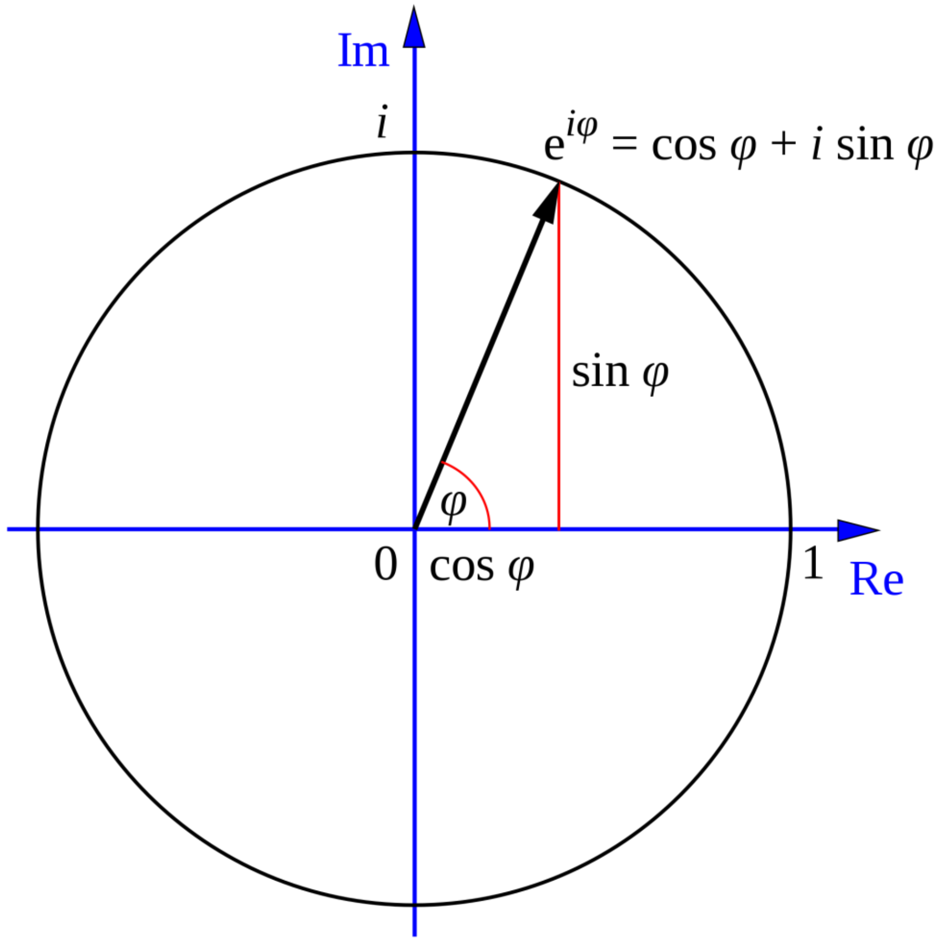
- $|(-1+i)^2| =$

- $Arg((-1+i)^2) =$

A: $\frac{3}{2}\pi$
B: $\frac{3}{4}\pi$
C: $\sqrt{2}$
D: 2
E: None of the above

Euler's Formula: $e^{i\theta} = \cos\theta + i\sin\theta$

- Real exponentials define exponential growth.
  - $e^0 = 1$
  - $e^1 = e \approx 2.718$
  - $e^2 \approx 7.389$
- Imaginary exponentials encode rotation around the complex origin.
  - $e^0 = 1$
  - $e^{\frac{\pi}{4}i} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$
  - $e^{\frac{\pi}{2}i} = i$
  - $e^{\pi i} = -1$



https://en.wikipedia.org/wiki/Euler%27s_formula#/media/File:Euler's_formula.svg

Polar form

- A complex number $z = a + bi$ can be rewritten as a scalar $|z|$ and an angle θ: $z = |z|(\cos\theta + i\sin\theta)$, where $|z| = a^2 + b^2$ and $\theta = Arg(z) = \begin{cases} \arctan\frac{b}{a}, & \text{if } a > 0 \\ \arctan\frac{b}{a} + \pi, & \text{if } a < 0 \end{cases}$.

- Complex exponential $e^z = e^{a+bi} = e^a e^{bi}$
$$= e^a(\cos b + i\sin b)$$
- Thus, $|e^z| = e^a$ and $Arg(e^z) = b$
- So, multiplying by a complex exponential scales by e^a and rotates by an angle b in radians.

Example

A: 0
B: 1
C: e
D: e^2
E: None of the above