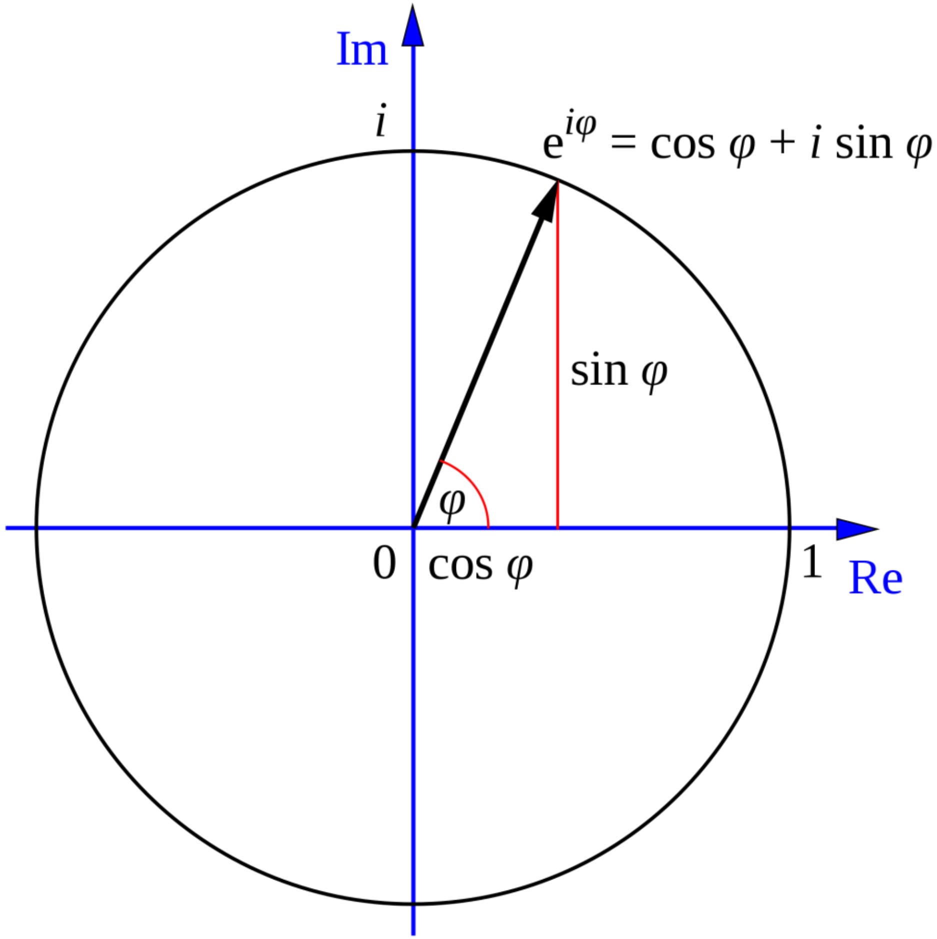
Complex eigenvalues and real solutions Lecture 9c: 2021-07-21

MAT A35 – Summer 2021 – UTSC

Prof. Yun William Yu

Euler's Formula: $e^{i\theta} = \cos\theta + i\sin\theta$

- Real exponentials define exponential growth.
  - $e^0 = 1$
  - $e^1 = e \approx 2.718$
  - $e^2 \approx 7.389$
- Imaginary exponentials encode rotation around the complex origin.
  - $e^0 = 1$
  - $e^{\frac{\pi}{4}i} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$
  - $e^{\frac{\pi}{2}i} = i$
  - $e^{\pi i} = -1$



https://en.wikipedia.org/wiki/Euler%27s_formula#/media/File:Euler's_formula.svg

Complex roots → Real solutions

- Consider the equation $y'' + y = 0$
- Use $e^{ix} = \cos x + i \sin x$

Char. poly. $\lambda^2 + 1 = 0$

$$\lambda = \pm\sqrt{-1} = \pm i$$

$$y = c_1 e^{ix} + c_2 e^{-ix}$$

$$e^{ix} = \cos x + i \sin x$$

$$e^{-ix} = \cos(-x) + i \sin(-x)$$

$$= \cos x - i \sin x$$

$$y = c_1 \cos x + i c_1 \sin x + c_2 \cos x - i c_2 \sin x$$

$$y = (c_1 + c_2) \cos x + i(c_1 - c_2) \sin x$$

Or $y = \hat{c}_1 \cos x + \hat{c}_2 \sin x$, $\hat{c}_1 = c_1 + c_2$

$\hat{c}_2 = i(c_1 - c_2)$

Another example

- $y'' + 2y' + 5y = 0$

$$\lambda^2 + 2\lambda + 5 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$$

$$y = c_1 \underline{e^{(-1+2i)x}} + c_2 \underline{e^{(-1-2i)x}}$$

$$e^{(-1+2i)x} = e^{-x}[\cos 2x + i \sin 2x]$$

$$e^{(-1-2i)x} = e^{-x}[\cos 2x - i \sin 2x]$$

So $e^{-x}\cos 2x$ and $e^{-x}\sin 2x$ are also independent solutions

OR

$$y = \hat{c}_1 e^{-x} \cos 2x + \hat{c}_2 e^{-x} \sin 2x$$

Complex roots with real coefficients

- Complex roots of a real polynomial always come in pairs $\boxed{a \pm ib}$.
- If a characteristic equation of an ODE has roots $a \pm ib$, then has complex solutions $e^{(a+ib)x}$ and $e^{(a-ib)x}$.
- Alternately, it has real solutions $e^{ax} \sin bx$ and $e^{ax} \cos bx$

Ex.

$$y'' - 2y' + 10y = 0$$

$$\lambda^2 - 2\lambda + 10 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4 - 40}}{2} = 1 \pm \sqrt{-9} = \boxed{1 \pm 3i}$$

$$y = c_1 e^x \sin 3x + c_2 e^x \cos 3x$$

Try it out

- Let $y'' + 4y' + 29y = 0$.
- Which of the following are solutions to the ODE?

$\lambda^2 + 4\lambda + 29 = 0$

$(\lambda+2)^2 + 25 = 0$

$(\lambda+2)^2 = -25$

$\lambda + 2 = \pm 5i$

$\lambda = -2 \pm 5i$

$e^{(-2+5i)x}, \quad e^{(-2-5i)x}$

$e^{-2x}\cos 5x, \quad e^{-2x}\sin 5x$

$\boxed{D}$

- What about real solutions?

$\boxed{C}$

$-\pi e^{-2x} e^{5ix} = -\pi e^{(-2x+5ix)}$

$= -\pi e^{(-2+5i)x}$

A: $e^{(-2+5i)x} + 4e^{(-2-5i)x}$
B: $-\pi e^{-2x} e^{5ix}$
C: $e^{-2x} \cos 5x$
D: All of the above
E: None of the above

Repeated complex eigenvalues of ODE

- Like repeated real roots, if $a \pm bi$ have multiplicity k, then $x^{k-1}e^{ax}\cos bx$ and $x^{k-1}e^{ax}\sin bx$ are solutions.

Ex.

$$y'''' + 8y'' + 16y = 0$$

$$\lambda^4 + 8\lambda^2 + 16 = 0$$

$$(\lambda^2 + 4)^2 = 0$$

$$(\lambda + 2i)^2(\lambda - 2i)^2 = 0$$

$$\lambda^2 = -4$$

$\lambda = \pm 2i$, multiplicity 2

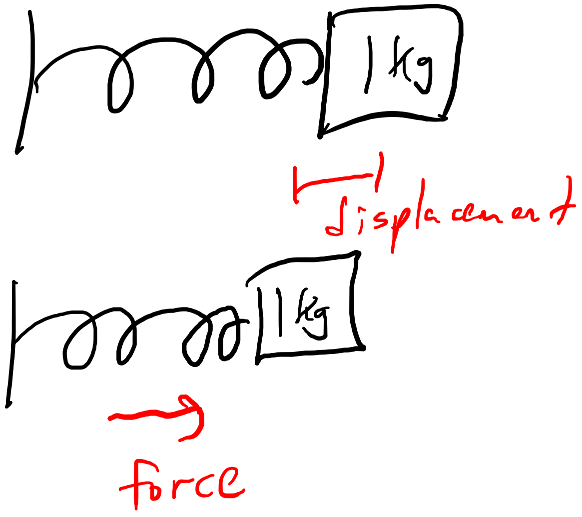
$$\Rightarrow y = c_1 \cos 2x + c_2 \sin 2x + c_3\, x \cos 2x + c_4\, x \sin 2x$$

Summary

- To solve a linear nth-order homogeneous ODE
$$a_n y^{(n)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0$$
- Construct the characteristic equation
$$a_n \lambda^n + \cdots + a_2 \lambda^2 + a_1 \lambda + a_0 = 0$$
- The n roots (counting multiplicity) of the characteristic equation are either real or come in complex conjugate pairs.
- If λ is a (real or complex) root of multiplicity k, then $e^{\lambda x}, xe^{\lambda x}, \ldots x^{k-1}e^{\lambda x}$ are k linearly independent solutions.
- If $\lambda = a \pm ib$ is a conjugate pair of complex roots, each of multiplicity k, then $e^{ax}\cos bx\,, xe^{ax}\cos bx\,, \ldots, x^{k-1}e^{ax}\cos bx$ and $e^{ax}\sin bx\,, xe^{ax}\sin bx\,, \ldots, x^{k-1}e^{ax}\sin bx$ are $2k$ linearly independent solutions.

Application: mass-spring system

- A spring acts on an attached 1kg object with force -4 Newtons/meter times the displacement in meters.
- Let y be the displacement of the object at time x, and y' is its velocity.
- By Newton's 2nd law, $F = ma$, where F is force, m is mass, and $a = y''$ is acceleration.

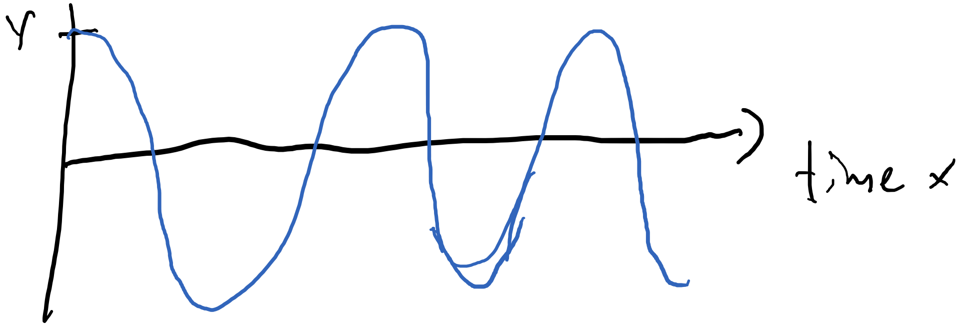


$$y'' = -4y$$

$$y'' + 4y = 0$$

$$\lambda^2 + 4 = 0$$

$$\lambda = \pm i$$



$$y = c_1 \cos 2x + c_2 \sin 2x$$

Initial Value Problem

- $y'' + 4y = 0$, where $y'(0) = 0$, $y(0) = 10$.

$y = c_1 \cos 2x + c_2 \sin 2x$

$y' = -2c_1 \sin 2x + 2c_2 \cos 2x$

$10 = y(0) = c_1 \cdot 1 + c_2 \cdot 0$

$\Rightarrow c_1 = 10$

$0 = y'(0) = -2c_1 \cdot 0 + 2c_2 \cdot 1$

$\Rightarrow c_2 = 0$

$\Rightarrow y = 10 \cos 2x$