Complex eigenvalues and real solutions Lecture 9c: 2021-07-21

MAT A35 – Summer 2021 – UTSC

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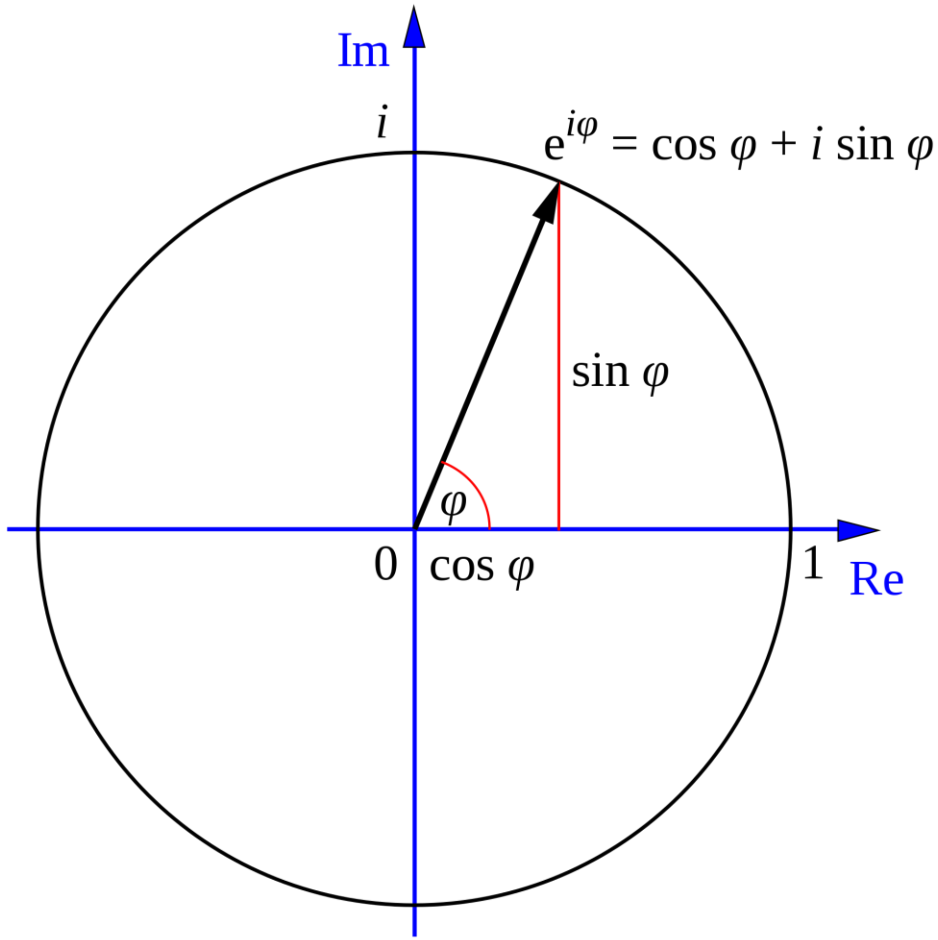
Euler's Formula: $e^{i\theta} = \cos\theta + i\sin\theta$

- Real exponentials define exponential growth.
  - $e^0 = 1$
  - $e^1 = e \approx 2.718$
  - $e^2 \approx 7.389$
- Imaginary exponentials encode rotation around the complex origin.
  - $e^0 = 1$
  - $e^{\frac{\pi}{4}i} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$
  - $e^{\frac{\pi}{2}i} = i$
  - $e^{\pi i} = -1$



https://en.wikipedia.org/wiki/Euler%27s_formula#/media/File:Euler's_formula.svg

Complex roots → Real solutions

- Consider the equation $y'' + y = 0$
- Use $e^{ix} = \cos x + i \sin x$

Another example

- $y'' + 2y' + 5y = 0$

Complex roots with real coefficients

- Complex roots of a real polynomial always come in pairs $a \pm ib$.
- If a characteristic equation of an ODE has roots $a \pm ib$, then has complex solutions $e^{(a+ib)x}$ and $e^{(a-ib)x}$.
- Alternately, it has real solutions $e^{ax} \sin bx$ and $e^{ax} \cos bx$

Try it out

- Let $y'' + 4y' + 29y = 0$.
- Which of the following are solutions to the ODE?

- What about real solutions?

A: $e^{(-2+5i)x} + 4e^{(-2-5i)x}$
B: $-\pi e^{-2x} e^{5ix}$
C: $e^{-2x}\cos 5x$
D: All of the above
E: None of the above

Repeated complex eigenvalues of ODE

- Like repeated real roots, if $a \pm bi$ have multiplicity k, then $x^{k-1}e^{ax}\cos bx$ and $x^{k-1}e^{ax}\sin bx$ are solutions.

Summary

- To solve a linear nth-order homogeneous ODE
$$a_n y^{(n)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0$$
- Construct the characteristic equation
$$a_n \lambda^n + \cdots + a_2 \lambda^2 + a_1 \lambda + a_0 = 0$$
- The n roots (counting multiplicity) of the characteristic equation are either real or come in complex conjugate pairs.
- If λ is a (real or complex) root of multiplicity k, then $e^{\lambda x}, xe^{\lambda x}, \ldots x^{k-1}e^{\lambda x}$ are k linearly independent solutions.
- If $\lambda = a \pm ib$ is a conjugate pair of complex roots, each of multiplicity k, then $e^{ax}\cos bx\,, xe^{ax}\cos bx\,, \ldots, x^{k-1}e^{ax}\cos bx$ and $e^{ax}\sin bx\,, xe^{ax}\sin bx\,, \ldots, x^{k-1}e^{ax}\sin bx$ are $2k$ linearly independent solutions.

Application: mass-spring system

- A spring acts on an attached 1kg object with force -4 Newtons/meter times the displacement in meters.
- Let y be the displacement of the object at time x, and y' is its velocity.
- By Newton's 2nd law, $F = ma$, where F is force, m is mass, and $a = y''$ is acceleration.

Initial Value Problem

- $y'' + 4y = 0$, where $y'(0) = 0$, $y(0) = 10$.