Nonhomogeneous constant coefficient ODEs
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(In)homogeneous constant coefficient linear ODEs

- Consider $a_n y^{(n)} + \cdots + a_1 y' + a_0 y = q(x)$, where a_i are constant coefficients and $q(x)$ is a functions of x.
  - If $q(x) = 0$, then *homogeneous.*
  - Otherwise, it is $\boxed{\textit{inhomogeneous}}$.

Ex.

$$\left.\begin{aligned} & y'' + 4y' + 5y = 5 \\ & y' + 7y = 3x \\ & y''' - y = 3e^{x} \end{aligned}\right\}$$

Solution to inhomogeneous problems

- Consider the inhomogeneous equation
$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = q(x)$$
- The associated homogeneous equation (which we know how to solve) is:
$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0$$
- If y_p is a any "particular" solution to the inhomogeneous equation, and y_h is the general solution to the associated homogeneous equation, then $y = y_p + y_h$ is the general solution to the inhomogeneous equation.

Example

- $y'' + 3y' + 2y = 6$

Homogeneous eq: $y'' + 3y' + 2y = 0$

$$\lambda^2 + 3\lambda + 2 = 0$$

$$(\lambda+1)(\lambda+2) = 0$$

$$\lambda = -1, -2$$

$$y_h = c_1 e^{-x} + c_2 e^{-2x}$$

Particular solution

Guess: $y_p = A$, A constant

$$y_p' = 0$$

$$y_p'' = 0$$

$$0 + 3 \cdot 0 + 2A = 6$$

$$\Rightarrow A = 3 \Rightarrow y_p = 3$$

$$y_{general} = y = y_h + y_p = c_1 e^{-x} + c_2 e^{-2x} + 3$$

Example

- $y'' + 3y' + 2y = e^{-3x}$

$y_h = c_1 e^{-x} + c_2 e^{-2x}$ (from last slide)

Guess: $y_p = A e^{-3x}$

$y_p' = -3Ae^{-3x}$

$y_p'' = 9Ae^{-3x}$

$y'' + 3y' + 2y = e^{-3x} \Rightarrow 9Ae^{-3x} + 3\cdot(-3Ae^{-3x}) + 2Ae^{-3x} = e^{-3x}$

$2Ae^{-3x} = e^{-3x}$

$2A = 1$

$A = \frac{1}{2}$

$y_p = \frac{1}{2} e^{-3x}$

$y_{gen} = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{2} e^{-3x}$

Method of undetermined coefficients

- Consider $a_n y^{(n)} + \cdots + a_1 y' + a_0 y = q(x)$
- Notice that whatever we guess for the particular solution y_p we have to take derivatives of it. A reasonable "Ansatz", guess, is y_p will "look like" the derivatives of $q(x)$ but with different coefficients.

Ex.

$q(x) = 5x^2 + 6x - 1$ $\qquad$ $y_p = Ax^2 + Bx + C$

$q(x) = e^{2x} + 2x^2$ $\qquad$ $y_p = Ae^{2x} + Bx^2 + Cx + D$

$q(x) = \sin x$ $\qquad$ $y_p = A\sin x + B\cos x$

Try it out: guess an Ansatz for y_p

- $q(x) = e^x + e^{2x}$

$Ae^x + Be^{2x}$

e^x $2e^{2x}$

A: Ae^x
B: Ae^{2x}
C: $Ae^x + Be^{2x}$
D: $Ae^x + Be^{2x} + C$
E: None of the above

- $q(x) = 3\boxed{x^2} + \boxed{\sin x}$

x^2 $\cos x$

$\boxed{x}$ $\sin x$

$\boxed{1}$

$\to 0$

$A\boxed{x^2} + B\boxed{x} + \boxed{C}$
$+ D\sin x + E\cos x$

A: $Ax^2 + B\sin x$
B: $Ax^2 + B\sin x + C\cos x$
C: $Ax^2 + Bx + C + D\sin x$
D: $Ax^2 + Bx + C + D\sin x + E\cos x$
E: None of the above

- $q(x) = \frac{1}{x}$

E, derivatives go on forever.

A: $A\ln x + B$
B: $\frac{A}{x} + B$
C: $\frac{A}{x} + \frac{B}{x^2} + D$
D: $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + D$
E: None of the above

Ansatz-homogeneous solution collisions

- What if your Ansatz looks like one of the homogeneous solutions?
- Then just like with repeated roots, will need to add an "x".

Ex. $y'' + 3y' + 2y = e^{-x}$

$y_h = c_1 e^{-x} + c_2 e^{-2x}$

Ansatz $y_p = Axe^{-x}$

need additional x

Ex. $y'' + 2y' + y = e^{-x}$

$y_h = c_1 e^{-x} + c_2 x e^{-x}$

Ansatz:

$\cancel{y_p = Ae^{-x}}$

$\cancel{y_p = Axe^{-x}}$

$y_p = Ax^2 e^{-x}$

Try it out: guess an Ansatz y_p

- $y'' + 3y' + 2y = e^x + e^{2x}$?

$\lambda^2 + 3\lambda + 2 = 0$

$(\lambda+1)(\lambda+2) = 0$

$\lambda = -1, -2$

$y_h = c_1 e^{-x} + c_2 e^{-2x}$

A: $Ae^x + Be^{2x}$
B: $Axe^x + Be^{2x}$
C: $Ae^x + Bxe^{2x}$
D: $Axe^x + Bxe^{2x}$
E: None of the above

- $y'' - y = e^x + e^{2x}$

$\lambda^2 - 1 = 0$

$\lambda = \pm 1$

$y_h = c_1 e^x + c_2 e^{-x}$

$Axe^x + Be^{2x}$

A: $Ae^x + Be^{2x}$
B: $Axe^x + Be^{2x}$
C: $Ae^x + Bxe^{2x}$
D: $Axe^x + Bxe^{2x}$
E: None of the above

- $y'' + y = \sin x$

$\lambda^2 + 1 = 0$

$\lambda = \pm i$

$y_h = c_1 \sin x + c_2 \cos x$

$Ax \sin x + Bx \cos x$

A: $A \sin x$
B: $A \sin x + B \cos x$
C: $Ax \sin x + B \cos x$
D: $Ax \sin x + Bx \cos x$
E: None of the above

Example

- $y' + 2y = x^2,\ y(0) = 1$

$\lambda + 2 = 0$

$\lambda = -2$

$y_h = c_1 e^{-2x}$

$y_p = Ax^2 + Bx + C$

$y_p' = 2Ax + B$

$y_p' + 2y = 2Ax + B + 2Ax^2 + 2Bx + 2C = x^2$

$$\left.\begin{aligned} 2Ax^2 &= x^2 \\ 2Ax + 2Bx &= 0 \\ B + 2C &= 0 \end{aligned}\right\} \quad \left.\begin{aligned} 2A &= 1 \\ A + B &= 0 \\ B + 2C &= 0 \end{aligned}\right\} \quad \begin{aligned} A &= \frac{1}{2} \\ B &= -\frac{1}{2} \\ C &= \frac{1}{4} \end{aligned}$$

$y_p = \frac{x^2}{2} - \frac{x}{2} + \frac{1}{4}$

$y_g = y_h + y_p = c_1 e^{-2x} + \frac{x^2}{2} - \frac{x}{2} + \frac{1}{4}$

$y(0) = 1 = c_1 + \frac{1}{4}$

$\Rightarrow c_1 = \frac{3}{4}$

$y = \frac{3}{4} e^{-2x} + \frac{x^2}{2} - \frac{x}{2} + \frac{1}{4}$

Summary

- Consider $a_n y^{(n)} + \cdots + a_1 y' + a_0 y = q(x)$
- We can compute the homogeneous solution by looking at roots of the characteristic polynomial $a_n \lambda^n + \cdots + a_1 \lambda + a_0 = 0$, and independent solutions will be of the form $e^{\lambda x}$ or $e^{Re(\lambda)x} \cos(Im(\lambda)x)$ and $e^{Re(\lambda)x} \sin(Im(\lambda)x)$.
- We can often guess a particular solution by using an Ansatz with undetermined coefficients that looks like the derivatives of $q(x)$. We can then solve for the coefficients.
- The general solution is then given by the homogeneous solution plus any particular solution.
- We can solve an initial value problem by plugging those values back into the general solution.