Nonhomogeneous constant coefficient ODEs
Lecture 9d: 2021-07-21

MAT A35 – Summer 2021 – UTSC

Prof. Yun William Yu

(In)homogeneous constant coefficient linear ODEs

- Consider $a_n y^{(n)} + \cdots + a_1 y' + a_0 y = q(x)$, where a_i are constant coefficients and $q(x)$ is a functions of x.
  - If $q(x) = 0$, then *homogeneous.*
  - Otherwise, it is *inhomogeneous*.

Solution to inhomogeneous problems

- Consider the inhomogeneous equation

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = q(x)$$

- The associated homogeneous equation (which we know how to solve) is:

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0$$

- If y_p is a any "particular" solution to the inhomogeneous equation, and y_h is the general solution to the associated homogeneous equation, then $y = y_p + y_g$ is the general solution to the inhomogeneous equation.

Example

- $y'' + 3y' + 2y = 6$

Example

- $y'' + 3y' + 2y = e^{-3x}$

Method of undetermined coefficients

- Consider $a_n y^{(n)} + \cdots + a_1 y' + a_0 y = q(x)$
- Notice that whatever we guess for the particular solution y_p we have to take derivatives of it. A reasonable "Ansatz", guess, is y_p will "look like" the derivatives of $q(x)$ but with different coefficients.

Try it out: guess an Ansatz

- $q(x) = e^x + e^{2x}$

A: Ae^x
B: Ae^{2x}
C: $Ae^x + Be^{2x}$
D: $Ae^x + Be^{2x} + C$
E: None of the above

- $q(x) = 3x^2 + \sin x$

A: $Ax^2 + B\sin x$
B: $Ax^2 + B\sin x + C\cos x$
C: $Ax^2 + Bx + C + D\sin x$
D:$Ax^2 + Bx + C + D\sin x + E\cos x$
E: None of the above

- $q(x) = \frac{1}{x}$

A: $A\ln x + B$
B: $\frac{A}{x} + B$
C: $\frac{A}{x} + \frac{B}{x^2} + D$
D: $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + D$
E: None of the above

Ansatz-homogeneous solution collisions

- What if your Ansatz looks like one of the homogeneous solutions?
- Then just like with repeated roots, will need to add an "x“.

Try it out: guess an Ansatz y_p

- $y'' + 3y' + 2y = e^x + e^{2x}$

A: $Ae^x + Be^{2x}$
B: $Axe^x + Be^{2x}$
C: $Ae^x + Bxe^{2x}$
D: $Axe^x + Bxe^{2x}$
E: None of the above

- $y'' - y = e^x + e^{2x}$

A: $Ae^x + Be^{2x}$
B: $Axe^x + Be^{2x}$
C: $Ae^x + Bxe^{2x}$
D: $Axe^x + Bxe^{2x}$
E: None of the above

- $y'' + y = \sin x$

A: $A \sin x$
B: $A \sin x + B \cos x$
C: $Ax \sin x + B \cos x$
D: $Ax \sin x + Bx \cos x$
E: None of the above

Example

- $y' + 2y = x^2, y(0) = 1$

Summary

- Consider $a_n y^{(n)} + \cdots + a_1 y' + a_0 y = q(x)$
- We can compute the homogeneous solution by looking at roots of the characteristic polynomial $a_n \lambda^n + \cdots + a_1 \lambda + a_0 = 0$, and independent solutions will be of the form $e^{\lambda x}$ or $e^{Re(\lambda)x} \cos(Im(\lambda)x)$ and $e^{Re(\lambda)x} \sin(Im(\lambda)x)$.
- We can often guess a particular solution by using an Ansatz with undetermined coefficients that looks like the derivatives of $q(x)$. We can then solve for the coefficients.
- The general solution is then given by the homogeneous solution plus any particular solution.
- We can solve an initial value problem by plugging those values back into the general solution.