tags: `MATA35-2021`

Quiz 2 - Practice - Solutions

Problem 1: Matrix operations [20pts]

Solve each of the following problems. If the answer is undefined, state so explicitly.

1.
$$\begin{bmatrix} 3 & -1 & 0 \\ 1 & 0 & 2 \\ -2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{bmatrix}$$

> Undefined since dimensions do not match.

2.
$$\begin{bmatrix} 3 & -1 & 0 \\ 1 & 0 & 2 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 7 \\ 7 & 5 \\ 1 & -5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 7 \\ 7 & 5 \\ 1 & -5 \end{bmatrix}$$

3.
$$\begin{bmatrix} 3 & -1 & 0 \\ 1 & 0 & 2 \\ -2 & 0 & 1 \end{bmatrix}^2 \begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 16 \\ 3 & -3 \\ -1 & -19 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 16 \\ 3 & -3 \\ -1 & -19 \end{bmatrix}$$

4.
$$\begin{bmatrix} 3 & -1 & 0 \\ 1 & 0 & 2 \\ -2 & 0 & 1 \end{bmatrix}^2 \begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{bmatrix}^2$$

> Undefined since dimensions don't match. (i.e. you cannot take the square of a non-square matrix)

5.
$$\left(\begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix} \right)^2$$

$$\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

Problem 2: Systems of linear equations [15pts]

Solve the following systems of equations. If there are multiple solutions, give both the most general form of the solution and at least one specific solution (without free variables). If there are no solutions, state as much.
Note: although I have given two problems on the practice quiz, there will only be one problems on the actual quiz.

1. $$\begin{aligned} w + x + 2y - z &= 1 \\ w + 2x - 2y - 3z &= 6 \\ 3w - x - y - z &= 6 \\ w + x + y - 2z &= 2 \end{aligned}$$

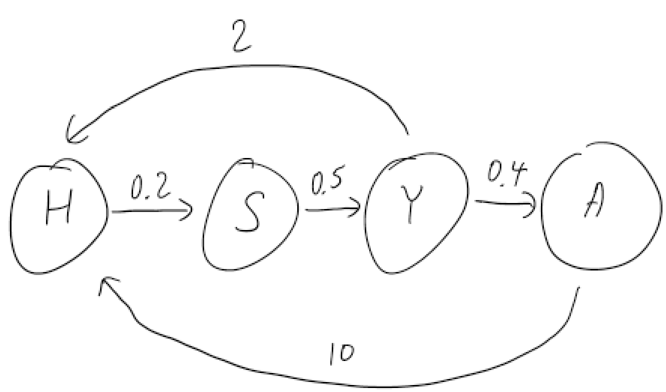
$w = 2, x = 1, y = -1, z = 0$

2. $$\begin{aligned} x + y - z &= 1 \\ x + 2y - 3z &= 3 \\ x + z &= -1 \\ y - 2z &= 2 \end{aligned}$$

Notice that the system is not linearly independent. Indeed, you can reduce down to an equivalent system with only 2 equations and 3 variables.
General solution $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 - z \\ 2 + 2z \\ z \end{bmatrix}$.
One specific solution $\begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}$ (setting $z = 1$)

Problem 3: Leslie matrix basic [15pts]

A population of sea turtles can be divided up into 4 stages, hatchlings (H), second-years (S), young adults (Y), and adults (A), with the Leslie diagram given below.



The population in year 1 is composed of 0 hatchlings, 10 second years, 5 young adults, and 2 adults.

1. Write the Leslie Matrix.

$$\begin{bmatrix} 0 & 0 & 2 & 10 \\ 0.2 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.4 & 0 \end{bmatrix}$$

2. Estimate the population in each group in year 2.

$$\begin{bmatrix} 0 & 0 & 2 & 10 \\ 0.2 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.4 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 10 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 30 \\ 0 \\ 5 \\ 2 \end{bmatrix}$$

3. Estimate the population in each group in year 3.

$$\begin{bmatrix} 0 & 0 & 2 & 10 \\ 0.2 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.4 & 0 \end{bmatrix}^2 \begin{bmatrix} 0 \\ 10 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 & 10 \\ 0.2 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.4 & 0 \end{bmatrix} \begin{bmatrix} 30 \\ 0 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 30 \\ 6 \\ 0 \\ 2 \end{bmatrix}$$

Note: I gave two separate problems 3 and 6 here about Leslie matrices in the practice quiz. On the actual quiz, I may combine them into a single problem.

Problem 4: Matrix inverses and determinants [15pts]

Find the determinant and multiplicative inverse for each of the following matrices. If an inverse does not exist or is undefined, say why.

Note: although I have given four problems on the practice quiz, there will only be two problems on the actual quiz.

1. $$\begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}$$

determinant = 1 and inverse is $\begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix}$

2. $$\begin{bmatrix} 3 & -1 & 0 \\ 1 & 0 & 2 \\ -1 & 1 & 2 \end{bmatrix}$$

determinant = -2 and inverse is $\begin{bmatrix} 1 & -1 & 1 \\ 2 & -3 & 3 \\ -\frac{1}{2} & 1 & -\frac{1}{2} \end{bmatrix}$

3. $$\begin{bmatrix} 3 & -1 \\ 1 & 0 \\ 2 & -1 \end{bmatrix}$$

determinant and inverse undefined since it is not a square matrix

4. $$\begin{bmatrix} 3 & -1 & 0 \\ 1 & 0 & 2 \\ -1 & 1 & 4 \end{bmatrix}$$

determinant $= 0$ so there exists no inverse.

Problem 5: Eigendecomposition [15pts]

Find all eigenvalues and eigenvalues for the following matrix. Show your work.

Note: although I have given two problems on the practice quiz, there will only be one problems on the actual quiz.

1.

$$\begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$$

$\lambda_1 = 1, v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\lambda_2 = 6, v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

2.

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$\lambda_1 = 3, v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

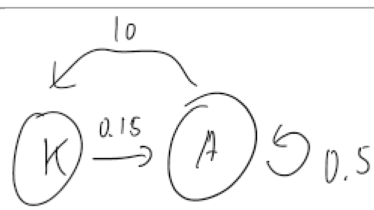
$\lambda_2 = -1, v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Problem 6: Leslie matrix eigenvectors [20pts]

A new invasive species of feral cats has just been accidentally released in Australia. As an ecologist, you have built a 2-stage Leslie population model for the cats, and have measured the following rates:

- Each year, an adult cat has on average 10 kittens.
- Each year, kittens survive to adulthood with probability 0.15 (if the kittens survive, they become adults the next year).
- Kittens don't have any children their first year of life.
- Each year, an adult has a 0.5 chance of dying.

1. Draw a Leslie diagram and Leslie matrix for this population.



$$\begin{bmatrix} 0 & 10 \\ 0.15 & 0.5 \end{bmatrix}$$

2. If you have measured the first year population to be 20 feral cats and no kittens, what is your estimate for the population after 5 years. You do not need to simplify, and can leave the answer in terms of fractions and powers.

We first compute the eigendecomposition:

$\lambda_1 = \frac{3}{2}, v_1 = \begin{bmatrix} 20 \\ 3 \end{bmatrix}$

$\lambda_2 = -1, v_1 = \begin{bmatrix} -10 \\ 1 \end{bmatrix}$

We then rewrite the population vector $\begin{bmatrix} 0 \\ 20 \end{bmatrix} = 4 \begin{bmatrix} 20 \\ 3 \end{bmatrix} + 8 \begin{bmatrix} -10 \\ 1 \end{bmatrix}$.

Then, $\begin{bmatrix} 0 & 10 \\ 0.15 & 0.5 \end{bmatrix}^5 \begin{bmatrix} 0 \\ 20 \end{bmatrix} 4 \cdot (1.5)^5 \begin{bmatrix} 20 \\ 3 \end{bmatrix} + 8 \cdot (-1)^5 \begin{bmatrix} -10 \\ 1 \end{bmatrix}$

3. What is the long-term ratio of adult cats to kittens?
   The long term ratio is 20 kittens to 3 adults because that is the ratio found in the eigenvector corresponding to the dominant eigenvalue (i.e. the eigenvalue with the largest absolute value, which in this case is 1.5).

Note: I gave two separate problems 3 and 6 here about Leslie matrices in the practice quiz. On the actual quiz, I may combine them into a single problem.