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Quiz 4 - Practice Problems

Problem 1: Classifying differential equations

Classify the following differential equations. Is it a partial differential equation, or just an ordinary differential equation?

If it is an ODE, answer the following: What is the order? Is it linear? Is it autonomous?

1. $y'' - 2y' - 3y = 3e^{2x}$

> ODE, 2nd-order, linear, nonautonomous

2. $\frac{dx}{dt} = \log(t - 2 + e^{\sin x})$

> ODE, 1st-order, nonlinear, nonautonomous

3. $y' + y^2 = 0$

> ODE, 1st-order, nonlinear, autonomous

4. $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 5$

> PDE

Problem 2: Solving a separable equation

Solve the following ODE:

$$y' = \frac{x-1}{\sin y}$$

> $(\sin y)dy = (x-1)dx$
> $-\cos y = 0.5x^2 - x + C$
> $y = \arccos(x - 0.5x^2 + C)$

Problem 3: Solving an exact equation

Solve the following ODE:

$$dx(2x \sin y + y) + dy(x^2 \cos y + x) = 0$$

> $f(x, y) = x^2 \sin y + xy + C$

Problem 4: Substitutions and integrating factors

It is straight-forward to solve first-order ODEs in two cases: (1) when they are separable, or (2) when they are exact differentials. In each of the following cases, find an appropriate substitution(s) or integrating factor to make the equation either separable or exact.

You do NOT need to solve the equations. You may state any change of variables as multiple substitutions. Integrating factors should be simplified as much as possible.

Integrating factor for linear 1st order:

- Given $y' + p(x)y = q(x)$
- $I(x) = \exp(\int p(x)dx)$
- If you multiply both sides by $I(x)dx$, and integrate both sides, you get $I(x)y = \int q(x)I(x)dx$, and can solve for y.

Common Substitution Guesses:

- $P(x, y)dx + Q(x, y)dy = 0$, where $P(tx, ty) = t^n P(x, y)$ and $Q(tx, ty) = t^n Q(x, y)$ for some integer n. Let $u = x/y$. Will get separable ODE.
- $(a_1 x + b_1 y + c_1)dx + (a_2 x + b_2 y + c_2)dy = 0$.
  - If the two lines are intersecting, then let $u = a_1 x + b_1 y + c_1$ and $v = a_2 x + b_2 y + c_2$. Will get case above, and need another substitution $z = u/v$ to get to separable.
  - If the two lines are parallel, then let $u = a_1 x + b_1 y + c_1$. Will get separable ODE.
- Bernoulli ODE: $y' + P(x)y = Q(x)y^n$. Multiply by $(1-n)y^{-n}$. Then let $u = y^{1-n}$. Will get 1st-order linear ODE. Then need to use Integrating Factor to make exact.

1. $(x + y - 1)dx + (x + y + 1)dy = 0$

> Sol 1: Two parallel lines. Let $u = x + y - 1$ to make separable.
> Sol 2: Exact equation already. Don't need an integrating factor.

2. $y' + 3x^2 y = \exp(-x^3)$

> 1st-order linear. Integrating factor $I(x) = e^{x^3}$

3. $(2x + y - 4)dx - (x - y + 1)dy = 0$

> Intersecting lines. Let $u = 2x + y - 4$ and $v = -x + y - 1$. Then let $z = u/v$ to make separable.
> Note: it is NOT exact.

4. $(y + t + 1)dt - (5y + 5t - 5)dy = 0$

> Parallel lines. Let $u = y + t + 1$ to make separable.

5. $\exp\left(\frac{x}{t}\right) dx + \tan\left(\frac{x}{t}\right) = 0$

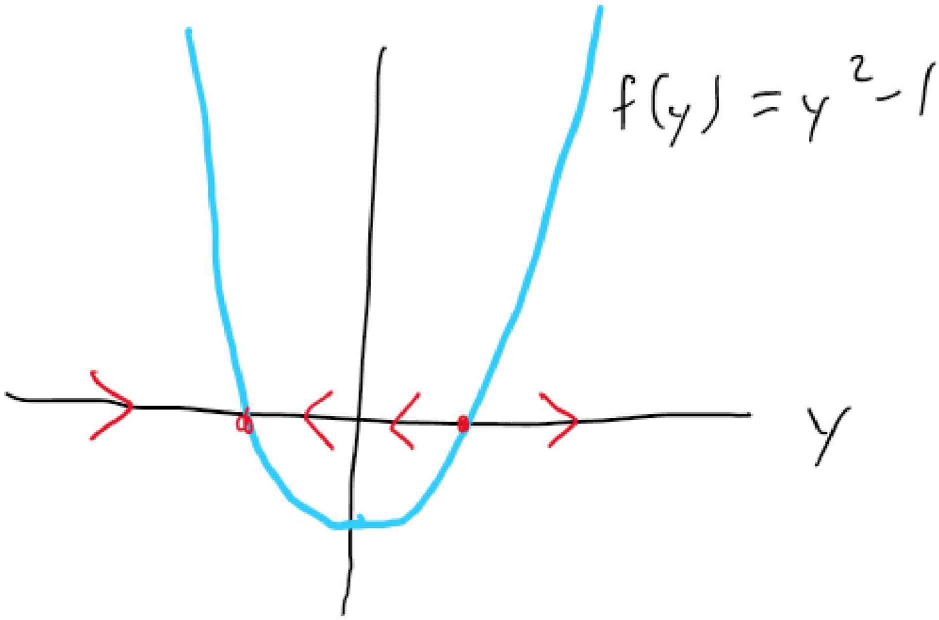
> Let $u = x/tu = x/t$ to make separable.

6. $y' = 4y + y^3$

> Bernoulli ODE. Rewrite $y' - 4y = y^3$.
> Multiply by $-2y^{-3}$ to get $-2y^{-3}y' + 8y^{-2} = -2$.
> Then let $u = y^{-2}$, $du = -2y^{-3}dy$, so $u' + 8u = -2$.
> Use integrating factor $I(x) = e^{8x}$ to make exact.

Problem 5: phase line and autonomous equation

Let $y' = y^2 - 1$.
Find the equilibria of this autonomous system.
Determine whether those equilibria are asymptotically stable, unstable, or semi-stable.



Equilibria: $y^2 - 1 = 0$

$\Rightarrow y = 1, -1.$

-1 is asymptotically stable

1 is unstable

Problem 6: Compartmentalized model

A tank contains 100 gallons of water. You accidentally dissolved 300 pounds of salt in that water, turning it into brine with 3 pounds salt / gallon water. However, you want the final solution to be 2 pounds salt per water.

1. You open up the drain at the bottom of the tank, which lets out 3 gallons of brine per minute from the tank, and simultaneously replace it with 3 gallons of brine with 1 pound salt per gallon. Assuming you stir constantly to keep the solution mixed, how long will it be before you achieve your desired salt concentation in the tank? (you may leave your answer in terms of e and $\ln$, as appropriate).

y = amt of salt

$y(0) = 300$ lb

$y' = 3 - 0.03y$

$y' + 0.03y = 3$

$IF = e^{0.03x}$

$ye^{0.03x} = \int 3e^{0.03x}\,dx = 100e^{0.03x} + C$

$y = 100 + Ce^{-0.03x}$

$y(0) = 300 = 100 + C$

$\Rightarrow C = 200$

$y = 100 + 200e^{-0.03x}$

Want $y(x) = 200$

$200 = 100 + 200e^{-0.03x}$

$1 = 2e^{-0.03x}$

$e^{-0.03x} = \frac{1}{2}$

$-0.03x = \ln\frac{1}{2} = -\ln 2$

$\boxed{x = \frac{100}{3}\ln 2 \text{ minutes}}$ $\approx$ 23.1 minutes

2. What if you mixed in 3 gallons of brine with 2 pounds salt per gallon? How long does it take for the solution to reach the desired concentation?

> It would take forever, because you would asymptotically approach 2 pounds of salt per gallon.

Problem 7: Euler's method

Let $y' = f(x, y) = x^2 + y^2$. Suppose you have initial conditions $y(0) = 1$. Use Euler's method to approximate the value at $y(3)$ with a step size of $\Delta x = 1$.

> $x_0 = 0$, $y_0 = 1$, $f(0, 1) = 1$
> $x_1 = 1$, $y_1 = y_0 + 1\Delta x = 2$, $f(1, 2) = 5$
> $x_2 = 2$, $y_2 = y_1 + 5\Delta x = 7$, $f(2, 7) = 53$
> $x_3 = 3$, $y_3 = y_2 + 53\Delta x = 60$.
> Thus, $y(3) \approx 60$.