tags: `MATA35-2021`

Quiz 4 - Thursday - Solutions

Problem 1: Classifying differential equations [16pts]

Classify the following differential equations. Is it a partial differential equation, or just an ordinary differential equation?

If it is an ODE, answer the following: What is the order? Is it linear? Is it autonomous?

1. $y''' + 3y' - \pi y = 4e^x$

> ODE, 3rd order, linear, nonautonomous

2. $y' = 4y^2 + 2x$

> ODE, 1st order, nonlinear, nonautonomous

3. $\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = 4z - x + y$

> PDE

4. $\dot{x} - 4x = 0$, where $\dot{x} = \frac{dx}{dt}$

> ODE, 1st order, linear, autonomous

Marking notes: 4pts each.

Problem 2: Solving an exact equation [14pts]

Solve the following ODE. Make sure that in addition to finding an appropriate $f(x, y)$ such that the left hand side is the total differential, you also give your answer as an implicit equation of x and y.

$$dx(y\cos(xy) - 2x) + dy(x\cos(xy) + 2) = 0$$

> $f(x, y) = \sin(xy) - x^2 + 2y + C$

Problem 3: Substitutions and integrating factors [20pts]

It is straight-forward to solve first-order ODEs in two cases: (1) when they are separable, or (2) when they are exact differentials. In each of the following cases, find an appropriate substitution(s) or integrating factor to make the equation either separable or exact.

You do NOT need to solve the equations. You may state any change of variables as multiple substitutions. Integrating factors should be simplified as much as possible.

Integrating factor for linear 1st order:

- Given $y' + p(x)y = q(x)$
- $I(x) = \exp(\int p(x)dx)$
- If you multiply both sides by $I(x)dx$, and integrate both sides, you get $I(x)y = \int q(x)I(x)dx$, and can solve for y.

Common Substitution Guesses:

- $P(x,y)dx + Q(x,y)dy = 0$, where $P(tx,ty) = t^n P(x,y)$ and $Q(tx,ty) = t^n Q(x,y)$ for some integer n. Let $u = x/y$. Will get separable ODE.
- $(a_1 x + b_1 y + c_1)dx + (a_2 x + b_2 y + c_2)dy = 0$.
    - If the two lines are intersecting, then let $u = a_1 x + b_1 y + c_1$ and $v = a_2 x + b_2 y + c_2$. Will get case above, and need another substitution $z = u/v$ to get to separable.
    - If the two lines are parallel, then let $u = a_1 x + b_1 y + c_1$. Will get separable ODE.
- Bernoulli ODE: $y' + P(x)y = Q(x)y^n$. Multiply by $(1-n)y^{-n}$. Then let $u = y^{1-n}$. Will get 1st-order linear ODE. Then need to use Integrating Factor to make exact.

1. $y' - 4x^3 y = \exp(x^4)$

> 1st-order linear. Integrating factor $I(x) = e^{-x^4}$.

2. $(2x - y - 10)dx + (x - 2y + 1)dy = 0$

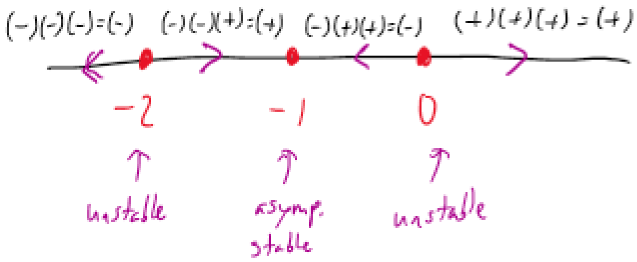
> Two intersecting lines. Let $u = 2x - y - 10$ and $v = x - 2y + 1$. Then let $z = u/v$ to make separable.

Problem 4: phase line and autonomous equation [15pts]

Let $y' = y(y+1)(y+2)$.
Find the equilibria of this autonomous system.
Determine whether those equilibria are asymptotically stable, unstable, or semi-stable.

$y' = y(y+1)(y+2)$

Equilibria: $y = -2, -1, 0$



Problem 5: Compartmentalized model [20pts]

A tank contains 100 gallons of water. You accidentally dissolved 200 pounds of salt in that water, turning it into brine with 2 pounds salt / gallon water. You open up the drain at the bottom of the tank, which lets out 10 gallons of water per minute from the tank, and simultaneously replace it with 10 gallons of fresh water per minute.

1. **[5pts]** Draw this as a one-compartment model.
2. **[5pts]** Write this as a differential equation.
3. **[10pts]** How long does it take to remove half the salt from the tank? (you may find it useful to know that $\ln 2 = 0.693$ and $\ln \frac{1}{2} = -0.693$

$\boxed{Y}$ $\quad y' = -\frac{y}{10}$

$\downarrow \frac{Y}{10}$ $\quad \frac{dy}{y} = -\frac{dx}{10}$

$$\ln|y| = -\frac{x}{10} + C$$

$$y = Ce^{-\frac{x}{10}}$$

$$y(0) = 200$$

$$\Rightarrow C = 200$$

$$y = 200e^{-\frac{x}{10}}$$

$$100 = 200e^{-\frac{x}{10}}$$

$$\frac{1}{2} = e^{-\frac{x}{10}}$$

$$-\frac{x}{10} = \ln\frac{1}{2}$$

$$x = -10\ln\frac{1}{2}$$

or

$$x = 10\ln 2 \approx 6.93 \text{ minutes}$$

Problem 6: Euler's method [15pts]

Let $y'=f(x,y)=x - y^2$. Suppose you have initial conditions $y(0)=1$. Use Euler's method to approximate the value at $y(3)$ with a step size of $\Delta x = 1$.

> $x_0 = 0$, $y_0 = 1$, $f(0,1)=-1$
> $x_1 = 1$, $y_1 = y_0 + 1 \Delta x = 0$, $f(1,0)=1$
> $x_2 = 2$, $y_2 = y_1 + 1 \Delta x = 1$, $f(2,1)=1$
> $x_3 = 3$, $y_3 = y_2 + 1 \Delta x = 2$.
> Thus, $y(3) \approx 2$.