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Quiz 5 - Practice Problems - Solutions

Problem 1: Higher-order inhomogeneous equation

Consider the following ODE:

$$y'''' + 2y''' + 2y'' + 2y' + y = 1 - e^x$$

1. Find the real homogeneous solution.
2. Find a real particular solution.
3. Find the real general solution.

> $\lambda^4 + 2\lambda^3 + 2\lambda^2 + 2\lambda + 1 = 0$
> $(\lambda+1)^2(\lambda^2+1) = 0$
> $\lambda = \begin{cases} -1, \text{ multiplicity } 2 \\ \pm i \end{cases}$
> Thus, the homogeneous solution is
> $y_h = c_1 e^{-x} + c_2 x e^{-x} + c_3 \sin x + c_4 \cos x$

> Ansatz for particular solution
> $y_p = A + Be^x$.
> $y_p' = Be^x$
> $y_p'' = Be^x$
> $y_p''' = Be^x$
> $y_p'''' = Be^x$
> Thus,
> $y'''' + 2y''' + 2y'' + 2y' + y = A + 8Be^x = 1 - e^x$
> So, $A = 1$ and $B = -\frac{1}{8}$
> $y_p = 1 - \frac{1}{8}e^x$

> The general solution is
> $y = c_1 e^{-x} + c_2 x e^{-x} + c_3 \sin x + c_4 \cos x + 1 - \frac{1}{8}e^x$

Problem 2: System of equations

Solve the following initial value problem:

$$\dot{x} = 2x - y + t$$
$$\dot{y} = -x + 2y$$

where $x(0) = \frac{13}{9}$ and $y(0) = -\frac{4}{9}$

Taking the derivative of the top equation,
$\ddot{x} = 2\dot{x} - \dot{y} + 1$
$\ddot{x} = 2\dot{x} - (-x + 2y) + 1 = \dot{x} + x - 2y + 1$
Note that from the top equation $y = 2x - \dot{x} + t$
$\ddot{x} = 2\dot{x} + x - 2(2x - \dot{x} + t) + 1$
$\ddot{x} = 4\dot{x} - 3x - 2t + 1$
$\ddot{x} - 4\dot{x} + 3x = -2t + 1$
Characteristic equation $\lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3) = 0$, so $\lambda = 1, 3$.
Homogeneous solution: $x_h = c_1 e^t + c_2 e^{3t}$.
The particular solution Ansatz is:
$x_p = At + B$
$\dot{x}_p = A$
$\ddot{x}_p = 0$
So $-4A + 3(At + B) = -2t + 1$
$3At + (3B - 4A) = -2t + 1$
Thus, $3A = -2$ and $3B - 4A = 1$
So $A = -\frac{2}{3}$, $B = -\frac{5}{9}$
Thus, $x_p = -\frac{2}{3}t - \frac{5}{9}$.
The general solution $x = x_h + x_p$, so
$x = c_1 e^t + c_2 e^{3t} - \frac{2}{3}t - \frac{5}{9}$
$\dot{x} = c_1 e^t + 3c_2 e^{3t} - \frac{2}{3}$
$y = 2x - \dot{x} + t$ from above, so
$y = 2c_1 e^t + 2c_2 e^{3t} - \frac{4}{3}t - \frac{10}{9} - c_1 e^t - 3c_2 e^{3t} + \frac{2}{3} + t$
$y = c_1 e^t - c_2 e^{3t} - \frac{1}{3}t - \frac{4}{9}$

If we plug in the initial values $x(0) = \frac{13}{9}$ and $y(0) = -\frac{4}{9}$, we find that $c_1 = 1$ and $c_2 = 1$.
Thus, the final solution to the initial value problem is:
$x = e^t + e^{3t} - \frac{2}{3}t - \frac{5}{9}$
$y = e^t - e^{3t} - \frac{1}{3}t - \frac{4}{9}$

Problem 3: Matrix equations

1. Given unknown functions $x(t)$ and $y(t)$, find the general solution to the system:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

> $$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{-5t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

2. Let $\dot{z} = Az$ for each of the following 2x2 matrices A. Classify the equilibrium at the origin by type and stability.
    - $\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$
    - $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$
    - $\begin{bmatrix} -3 & 1 \\ 0 & -1 \end{bmatrix}$
    - $\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$

> saddle point, unstable.
> center, stable.
> node, asymptotically stable.
> spiral, unstable

Problem 4: Word problem

Chemicals enter a house's basement air at a rate of 0.1 mg per minute. Let $F(t)$ and $B(t)$ denote the total amount of chemical present in the first-story air and the basement air after t minutes respectively.

Both the first floor and the basement have volumes of 200 m^3 each. Air flows from the basement into the first floor at a rate of 2 m^3 per minute, while air flows from the first floor to the outside at the rate of 4 m^3 per minute. Air from the outside (with no chemicals present) replenishes the air in both rooms to keep the volumes constant.

1. Draw a 2-compartment model for B and F.
2. Write a system of two first-order differential equations modelling the system.
3. Find the equilibrium values for B and F. (Recall: if a system is at its equilibrium values, then there is no change over time in any of its variables.)

$$\begin{cases} \dot{B} = -0.01B + 0.1 \\ \dot{F} = 0.01B - 0.02F \end{cases}$$

Equilibria

$$0 = \dot{B} = -0.01B + 0.1$$

$$0 = \dot{F} = 0.01B - 0.02F$$

$$\Rightarrow \quad 0.01B = 0.1$$

$$\Rightarrow \quad B = 10 \text{ mg}$$

$$\Rightarrow \quad 0 = B - 2F$$

$$\Rightarrow \quad F = 5 \text{ mg}$$