tags: `MATA35-2021`

Quiz 5 - Thursday - Solutions

Problem 1: Higher-order inhomogeneous equation [25pts]

Consider the following ODE:

$$y''' + y'' + y' + y = 4 + 8e^x$$

1. **[10pts]** Find the real homogeneous solution.
2. **[10pts]** Find a real particular solution.
3. **[5pts]** Find the real general solution.

> $\lambda^3 + \lambda^2 + \lambda + 1 = 0$
> $(\lambda + 1)(\lambda^2 + 1) = 0$
> $\lambda = \begin{cases} -1, \\ \pm i \end{cases}$
> Thus, the homogeneous solution is
> $y_h = c_1 e^{-x} + c_2 \sin x + c_3 \cos x$

> Ansatz for particular solution
> $y_p = A + Be^x.$
> $y_p' = Be^x$
> $y_p'' = Be^x$
> $y_p''' = Be^x$
> Thus,
> $y''' + y'' + y' + y = A + 4Be^x = 4 + 8e^x$
> So, $A = 4$ and $B = 2$
> $y_p = 4 + 2e^x$

> The general solution is
> $y = c_1 e^{-x} + c_2 \sin x + c_3 \cos x + 4 + 2e^x$

Problem 2: System of equations [25pts]

Solve the following initial value problem:

$$\dot{x} = x + 3y$$
$$\dot{y} = x + 3y$$

where $x(0) = 5$ and $y(0) = 1$.
Hint: You learned two different methods for solving this problem, and you may use whichever one is easier.

Marking notes:

- Getting to the correct characteristic equation is worth 5pts. (whether the student used substitution or the determinant)
- Getting the correct general solution for x is worth 5pts.
- Getting the correct general solution for y is worth 5pts. (note that if the student solved it as a matrix equation, then they should get both x and y simultaneously)
- Having the right setup for the initial value problem is worth 5 pts.
- Actually solving the initial value problem to get the correct final solutions for x and y is worth 5 pts.

Taking the derivative of the top equation,
$\ddot{x} = \dot{x} + 3\dot{y}$
$\ddot{x} = \dot{x} + 3(x + 3y) = \dot{x} + 3x + 9y$
Note that from the top equation $y = \frac{1}{3}(\dot{x} - x)$
$\ddot{x} = \dot{x} + 3x + 3(\dot{x} - x) = 4\dot{x}$
Thus, $\ddot{x} - 4\dot{x} = 0$.
Characteristic equation $\lambda^2 - 4\lambda = \lambda(\lambda - 4) = 0$, so $\lambda = 0, 4$.
The equation is homogeneous, so the general solution is
$x = c_1 e^0 t + c_2 e^{4t} = c_1 + c_2 e^{4t}$
$\dot{x} = 4c_2 e^{4t}$
$y = \frac{1}{3}(\dot{x} - x)$ from above, so
$y = \frac{1}{3}(4c_2 e^{5t} - (c_1 + c_2 e^{4t})) = \frac{1}{3}(3c_2 e^{5t} - c_1) = c_2 e^{5t} - \frac{1}{3}c_1$
$y = -\frac{1}{3}c_1 + c_2 e^{5t}$

If we plug in the initial values $x(0) = 5$ and $y(0) = 1$, we find that $c_1 = 3$ and $c_2 = 2$.
Thus, the final solution to the initial value problem is:
$x = 3 + 2e^{4t}$
$y = -1 + 2e^{4t}$

Alternately, we can rewrite as a matrix equation

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

The eigendecomposition of the matrix has eigenvalues 0, 5.

$\lambda_1 = 0, v_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

$\lambda_2 = 4, v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Thus,

$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} = \begin{bmatrix} -3c_1 + c_2 e^{4t} \\ c_1 + c_2 e^{4t} \end{bmatrix}$

Plugging into $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$, we get $c_1 = -1$, $c_2 = 2$

Thus, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 + 2e^{4t} \\ -1 + 2e^{4t} \end{bmatrix}$

Problem 3: Matrix equations [24pts + 3pts bonus]

Let $\dot{z} = Az$ for each of the following 2x2 matrices A. Classify the equilibrium at the origin by type and stability. **Bonus**: if the type is a node, further specify if it is proper, improper, or neither.

1. $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$
2. $\begin{bmatrix} -1 & -1 \\ 9 & -1 \end{bmatrix}$
3. $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$
4. $\begin{bmatrix} 0 & 4 \\ -1 & 0 \end{bmatrix}$

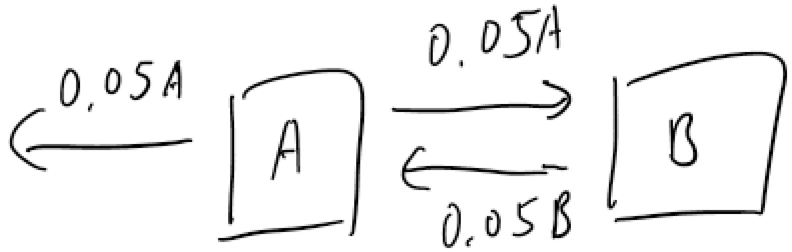
Marking notes: 3pts for getting the type correct, and 3pts for getting the stability correct. Note that for stability, “stable” and “asymptotically stable” are different things. The 3pt bonus is for correctly stating the proper node.
saddle point, unstable.
spiral, asymptotically stable
proper node, unstable
center, stable

Problem 4: Word problem

Tank A contains 100 gallons of pure water. Tank B contains 200 gallons of water with 50 lb of salt dissolved. Water is pumped from tank A to tank B at a rate of 5 gallons per minute. Water is also drained from tank A to the outside at a rate of 5 gallons per minute. Water is pumped from tank B to tank A at a rate of 10 gallons per minute. Pure water is added to tank B to keep the total volume constant.

Let $A(t)$ and $B(t)$ denote the total amount of salt present in tanks A and B respectively.

1. **[10pts]** Draw a 2-compartment model for A and B.
2. **[10pts]** Write a system of two first-order differential equations modelling the system.
3. **[6pts]** Find the equilibrium values for A and B.



$$\dot{A} = -0.1A + 0.05B$$

$$\dot{B} = 0.05A - 0.05B$$

The equilibrium values are $A = 0$ and $B = 0$.