Systems of Linear ODEs Lecture 10a: 2023-03-20

MAT A35 – Winter 2023 – UTSC

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System of two first-order ODEs

- Let t be the independent time variable, and let x and y be two dependent variables.
- If a, b, c, d are constants, then

$$\begin{cases} \dfrac{dx}{dt} = ax + by \\ \dfrac{dy}{dt} = cx + dy \end{cases}$$

$\dot{x} = \frac{dx}{dt}$

$\dot{y} = \frac{dy}{dt}$

is a homogeneous system of two first-order linear ODEs.

- Similarly, $\begin{cases} \dfrac{dx}{dt} = ax + by + f(t) \\ \dfrac{dy}{dt} = cx + dy + g(t) \end{cases}$

is an inhomogeneous system of two first-order linear ODEs.

Reduction method

- We can solve a system of two first-order linear ODEs by converting it into a single second-order linear ODE.
  - First compute $\ddot{x}$ as a function of $x, y, \dot{y}$ from the first equation.
  - Then eliminate $\dot{y}$ by substituting in the second equation.
  - Then, eliminate y by substituting in the first equation.

Ex.

$$\overbrace{\frac{dx}{dt}:\ \dot{x} = x + 2y}^{y = \frac{\dot{x} - x}{2}}$$

$$\frac{dy}{dt}\quad \dot{y} = 4x - y$$

$$\frac{d}{dt}\frac{dx}{dt} = \frac{d^2x}{dt^2} = \ddot{x} = \dot{x} + 2\dot{y}$$

$$\ddot{x} = \dot{x} + 2(4x - y)$$

$$\ddot{x} = \dot{x} + 8x - 2y$$

$$\ddot{x} = \dot{x} + 8x - 2\left(\frac{\dot{x} - x}{2}\right)$$

$$\ddot{x} = \dot{x} + 8x - \dot{x} + x$$

$$\ddot{x} = 9x$$

$$\Rightarrow \quad \ddot{x} - 9x = 0$$

Solve 2nd-order ODE

- Find eigenvalues/roots of the characteristic equation.
- Solve for x.
- Solve for y.

$$\left\{\begin{array}{l} \ddot{x} - 9x = 0 \\ \lambda^2 - 9 = 0 \\ \lambda = \pm 3 \end{array}\right.$$

$$x(t) = c_1 e^{3t} + c_2 e^{-3t}$$

$$\dot{x}(t) = 3c_1 e^{3t} - 3c_2 e^{-3t}$$

$$y = \frac{\dot{x} - x}{2}$$

$$y = \frac{1}{2}\left[\overbrace{3c_1 e^{3t} - 3c_2 e^{-3t}}^{\dot{x}} \overbrace{- c_1 e^{3t} - c_2 e^{-3t}}^{-x}\right]$$

$$\boxed{x = c_1 e^{3t} + c_2 e^{-3t}}$$

$$\boxed{y = c_1 e^{3t} - 2c_2 e^{-3t}}$$

Initial values

- If there are initial values, plug them in.

Ex. $x = c_1 e^{3t} + c_2 e^{-3t}$ $\quad y = c_1 e^{3t} - 2c_2 e^{-3t}$

IVP: $x(0) = 2, \quad y(0) = -1$

$t = 0$

$2 = x = c_1 + c_2$ $\quad -1 = y = c_1 - 2c_2$

$$\begin{cases} c_1 + c_2 = 2 \\ c_1 - 2c_2 = -1 \end{cases}$$

$3c_2 = 3$

$c_2 = 1$

$c_1 = 1$

$\Rightarrow$ $x = e^{3t} + e^{-3t}$

$y = e^{3t} - 2e^{-3t}$

Try it out

$y = \dot{x}$

comma

- $\begin{cases} \dot{x} = y \\ \dot{y} = -x + 2y \end{cases}, \; x(0) = 1, y(0) = 3$

A: $\ddot{x} = 0$
B: $\ddot{x} = 1$
C: $\ddot{x} = y$
D: $\ddot{x} = \dot{y}$
E: None of the above

- Step 1: Find $\ddot{x}$.

$$\ddot{x} = \frac{d}{dt}\frac{dx}{dt} = \frac{d}{dt} y = \dot{y}$$

- Step 2: Get rid of $\dot{y}$.

$$\ddot{x} = \dot{y} = -x + 2y$$

A: $\ddot{x} = y$
B: $\ddot{x} = -x + 2y$
C: $\ddot{x} = -xy + 2y^2$
D: $\ddot{x} = \dot{y}$
E: None of the above

- Step 3: Get rid of y and rewrite.

$$\ddot{x} = -x + 2y$$
$$\ddot{x} = -x + 2\dot{x}$$
$$\ddot{x} - 2\dot{x} + x = 0$$

A: $\ddot{x} + 2\dot{x} + x = 0$
B: $\ddot{x} - 2\dot{x} + x = 0$
C: $\ddot{x} + 2\dot{x} - x = 0$
D: $\ddot{x} - 2\dot{x} - x = 0$
E: None of the above

Try it out (continued)

- Step 4: Solve the ODE

$$\ddot{x} - 2\dot{x} + x = 0$$

$$\lambda^2 - 2\lambda + 1 = 0$$

$$(\lambda - 1)^2 = 0$$

$\lambda = 1$, mult 2

$$\Rightarrow x = c_1 e^t + c_2 t e^t$$

A: $x = c_1 e^t + c_2 e^{-t}$
B: $x = c_1 e^t + c_2 e^{2t}$
C: $x = c_1 e^t + c_2 x e^t$
D: $x = c_1 e^t + c_2 t e^t$
E: None of the above

- Step 5: Plug back in to solve for y.

$y = \dot{x}$

$$\dot{x} = y = c_1 e^t + c_2 e^t + c_2 t e^t$$

$$c_2 \frac{d}{dt}\left[t e^t \right] = c_2 \left[t e^t + e^t \right]$$

$$y = (c_1 + c_2) e^t + c_2 t e^t$$

A: $y = c_1 e^t + c_2 t e^t$
B: $y = (c_1 + c_2) e^t + c_2 t e^t$
C: $y = c_1 e^t + (c_1 + c_2) t e^t$
D: $y = (c_1 + c_2) e^t + (c_2 - c_2) t e^t$
E: None of the above

Try it out (continued)

t

- Plug in initial values $x(0) = 1, y(0) = 3$

$x = c_1 e^t + c_2 te^t$

$y = (c_1 + c_2) e^t + c_2 te^t$

$1 = x(0) = c_1 e^0 + c_2 \cdot 0 \cdot e^0 = c_1$

$\Rightarrow c_1 = 1$

$3 = y(0) = c_1 + c_2$

$\Rightarrow c_2 = 2$

$$\boxed{\begin{cases} x = e^t + 2te^t \\ y = 3e^t + 2te^t \end{cases}}$$

A: $c_1 = 1, c_2 = 1$

B: $c_1 = 1, c_2 = 2$

C: $c_1 = 2, c_2 = 1$

D: $c_1 = 2, c_2 = 2$

E: None of the above

Inhomogeneous example

$$\dot{x} = x + y + 9t$$
$$\frac{dx}{dt} = x + y + 9t$$
$$\frac{d^2x}{dt^2} = \frac{d}{dt}x + \frac{d}{dt}y + \frac{d}{dt}[9t]$$
$$\ddot{x} = \dot{x} + \dot{y} + 9$$

- $\begin{cases} \dot{x} = x + y + 9t \\ \dot{y} = 4x + y + 3 \end{cases}$

$y = \dot{x} - x - 9t$

$\cancel{y = \dot{y} - 4x - 3}$

Take $\boxed{\frac{d}{dt}\dot{x}}$

1. $\ddot{x} = \dot{x} + \dot{y} + 9$

Substitute in $\dot{y}$

2. $\ddot{x} = \dot{x} + (4x + y + 3) + 9$

$\ddot{x} = \dot{x} + 4x + y + 12$

Substitute in y

3. $\ddot{x} = \dot{x} + 4x + (\dot{x} - x - 9t) + 12$

$\ddot{x} = 2\dot{x} + 3x - 9t + 12$

$\ddot{x} - 2\dot{x} - 3x = -9t + 12$

Rewrite

Example (continued)

- $\ddot{x} - 2\dot{x} - 3x = \underline{-9t} + \underline{12}$

$\ddot{x} - 2\dot{x} - 3x = 0$

$\lambda^2 - 2\lambda - 3 = 0$

$(\lambda - 3)(\lambda + 1) = 0$

$\lambda = -1,\ 3$ eigs

$x_h = c_1 e^{-t} + c_2 e^{3t}$

Ansatz $x_p = At + B$

$\dot{x}_p = A$

$\ddot{x}_p = 0$

- $0 - 2A - 3(At + B) = -9t + 12$

$-3At + (-2A - 3B) = -9t + 12$

$$\left.\begin{aligned} -3A &= -9 \\ -2A - 3B &= 12 \end{aligned}\right\} \quad \begin{aligned} A &= 3 \\ B &= -6 \end{aligned}$$

$\Rightarrow\ x_p = 3t - 6$

$x_{gen} = x_h + x_p$

$$\boxed{x_{gen} = c_1 e^{-t} + c_2 e^{3t} + 3t - 6}$$

Solve for y

- $\begin{cases} \dot{x} = x + y + 9t \\ \dot{y} = 4x + y + 3 \end{cases}$

$y = \dot{x} - x - 9t$

- $x = c_1 e^{-t} + c_2 e^{3t} + 3t - 6$

$\dot{x} = -c_1 e^{-t} + 3c_2 e^{3t} + 3$

$y = -c_1 e^{-t} + 3c_2 e^{3t} + 3 - (c_1 e^{-t} + c_2 e^{3t} + 3t - 6) - 9t$

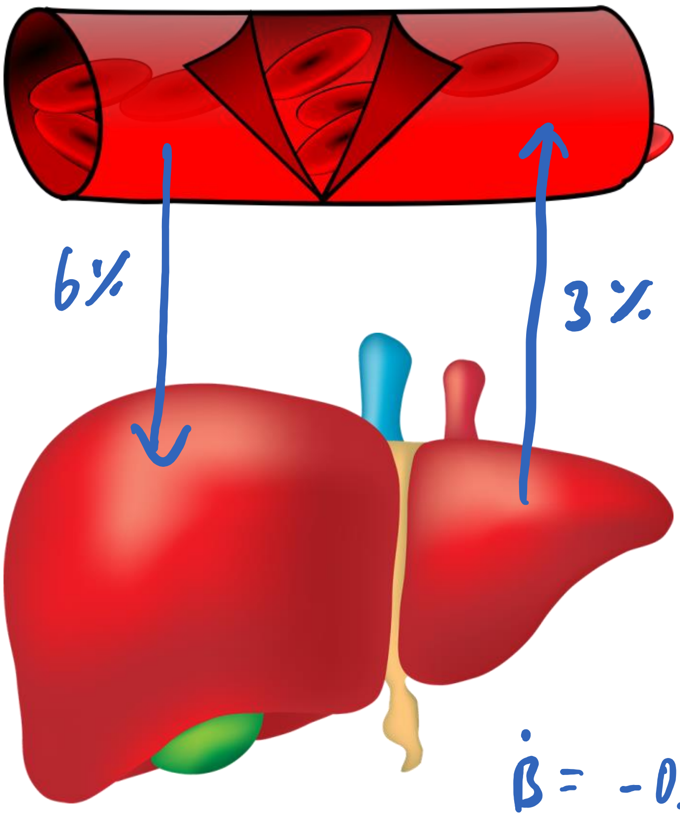
$y = -2c_1 e^{-t} + 2c_2 e^{3t} - 12t + 9$

Application (Bittinger, 9.3, Ex. 5)

- 3mg of glactosyl human serum albumin (Tc-GSA) is injected into a patient's bloodstream to measure liver function. After injection, Tc-GSA is transferred from the blood to the liver at 6% per minute, and from the liver into blood at 3% per minute.
- How much Tc-GSA is in the blood or liver as a function of time?

$B(0) = 3mg$

$L(0) = 0mg$



$$\begin{tikzcd} \boxed{B} \arrow[d, shift right, "0.06B"'] \\ \boxed{L} \arrow[u, shift right, "0.03L"'] \end{tikzcd}$$

$$\dot{B} = -0.06B + 0.03L$$

$$\dot{L} = 0.06B - 0.03L$$

Application (continued)

- $\begin{cases} \dot{B} = -0.06B + 0.03L \\ \dot{L} = \ \ 0.06B - 0.03L \end{cases}$

$0.03L = \dot{B} + 0.06B$

$3L = 100\dot{B} + 6B$

$L = \frac{100}{3}\dot{B} + 2B$

$\ddot{B} = -0.06\dot{B} + 0.03\dot{L}$

$\ddot{B} = -0.06\dot{B} + 0.03[0.06B - 0.03L]$

$\ddot{B} = -0.06\dot{B} + 0.0018B - 0.0009L$

- $\ddot{B} = -0.06\dot{B} + 0.0018B - 0.0009\left[\frac{100}{3}\dot{B} + 2B\right]$

$= -0.06\dot{B} + 0.0018B - 0.03\dot{B} - 0.0018B$

$\ddot{B} = -0.09\dot{B}$

$\ddot{B} + 0.09\dot{B} = 0$

Application (continued)

- $\ddot{B} + 0.09\dot{B} = 0$

$$L = \frac{100}{3}\dot{B} + 2B$$

$$\lambda^2 + 0.09\lambda = 0$$

$$\lambda(\lambda + 0.09) = 0$$

$$\lambda = 0, -0.09$$

$$B = c_1 e^{0t} + c_2 e^{-0.09t}$$

$$B = c_1 + c_2 e^{-0.09t}$$

$$\dot{B} = -0.09\, c_2 e^{-0.09t}$$

$$L = -3c_2 e^{-0.09t} + 2c_1 + 2c_2 e^{-0.09t}$$

$$L = 2c_1 - c_2 e^{-0.09t}$$

Initial Value Problem

- $\begin{cases} B(t) = c_1 + c_2 e^{-0.09t} \\ L(t) = 2c_1 - c_2 e^{-0.09t} \end{cases}$, with initial values $\begin{cases} B(0) = 3 \\ L(0) = 0 \end{cases}$

$3 = B(0) = c_1 + c_2 e^0 = c_1 + c_2$

$0 = L(0) = 2c_1 - c_2 e^0 = 2c_1 - c_2$

$\Rightarrow 3 = 3c_1$

$c_1 = 1$

$\Rightarrow c_2 = 2$

$B(t) = 1 + 2e^{-0.09t}$

$L(t) = 2 - 2e^{-0.09t}$

$\lim_{t\to\infty} B(t) = 1 \text{ mg}$

$\lim_{t\to\infty} L(t) = 2 \text{ mg}$