Systems of Linear ODEs
Lecture 10a: 2023-03-20

MAT A35 – Winter 2023 – UTSC

Prof. Yun William Yu

System of two first-order ODEs

- Let t be the independent time variable, and let x and y be two dependent variables.
- If a, b, c, d are constants, then
$$\begin{cases} \dfrac{dx}{dt} = ax + by \\ \dfrac{dy}{dt} = cx + dy \end{cases}$$
is a homogeneous system of two first-order linear ODEs.
- Similarly, $\begin{cases} \dfrac{dx}{dt} = ax + by + f(t) \\ \dfrac{dy}{dt} = cx + dy + g(t) \end{cases}$
is an inhomogeneous system of two first-order linear ODEs.

Reduction method

- We can solve a system of two first-order linear ODEs by converting it into a single second-order linear ODE.
  - First compute $\ddot{x}$ as a function of $x, y, \dot{y}$ from the first equation.
  - Then eliminate $\dot{y}$ by substituting in the second equation.
  - Then, eliminate y by substituting in the first equation.

Solve 2^{nd}-order ODE

- Find eigenvalues/roots of the characteristic equation.
- Solve for x.
- Solve for y.

Initial values

- If there are initial values, plug them in.

Try it out

- $\begin{cases} \dot{x} = y \\ \dot{y} = -x + 2y \end{cases}, x(0) = 1, y(0) = 3$
- Step 1: Find $\ddot{x}$.

A: $\ddot{x} = 0$
B: $\ddot{x} = 1$
C: $\ddot{x} = y$
D: $\ddot{x} = \dot{y}$
E: None of the above

- Step 2: Get rid of $\dot{y}$.

A: $\ddot{x} = y$
B: $\ddot{x} = -x + 2y$
C: $\ddot{x} = -xy + 2y^2$
D: $\ddot{x} = \dot{y}$
E: None of the above

- Step 3: Get rid of y and rewrite.

A: $\ddot{x} + 2\dot{x} + x = 0$
B: $\ddot{x} - 2\dot{x} + x = 0$
C: $\ddot{x} + 2\dot{x} - x = 0$
D: $\ddot{x} - 2\dot{x} - x = 0$
E: None of the above

Try it out (continued)

- Step 4: Solve the ODE

A: $x = c_1 e^t + c_2 e^{-t}$
B: $x = c_1 e^t + c_2 e^{2t}$
C: $x = c_1 e^t + c_2 x e^t$
D: $x = c_1 e^t + c_2 t e^t$
E: None of the above

- Step 5: Plug back in to solve for y.

A: $y = c_1 e^t + c_2 t e^t$
B: $y = (c_1 + c_2) e^t + c_2 t e^t$
C: $y = c_1 e^t + (c_1 + c_2) t e^t$
D: $y = (c_1 + c_2) e^t + (c_2 - c_2) t e^t$
E: None of the above

Try it out (continued)

- Plug in initial values $x(0) = 1, y(0) = 3$

A: $c_1 = 1, c_2 = 1$
B: $c_1 = 1, c_2 = 2$
C: $c_1 = 2, c_2 = 1$
D: $c_1 = 2, c_2 = 2$
E: None of the above

Inhomogeneous example

- $\begin{cases} \dot{x} = x + y + 9t \\ \dot{y} = 4x + y + 3 \end{cases}$

Example (continued)

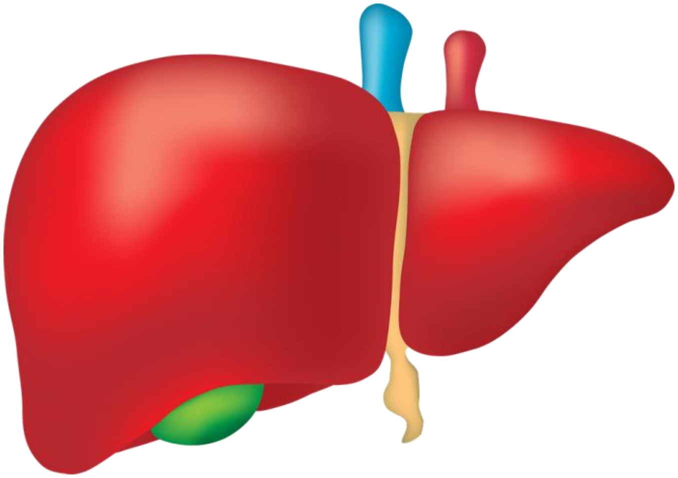
- $\ddot{x} - 2\dot{x} - 3x = -9t + 12$

Solve for y

- $\begin{cases} \dot{x} = x + y + 9t \\ \dot{y} = 4x + y + 3 \end{cases}$
- $x = c_1 e^{-t} + c_2 e^{3t} + 3t - 6$

Application (Bittinger, 9.3, Ex. 5)

- 3mg of glactosyl human serum albumin (Tc-GSA) is injected into a patient's bloodstream to measure liver function. After injection, Tc-GSA is transferred from the blood to the liver at 6% per minute, and from the liver into blood at 3% per minute.
- How much Tc-GSA is in the blood or liver as a function of time?



Application (continued)

- $\begin{cases} \dot{B} = -0.06B + 0.03L \\ \dot{L} = \ \ 0.06B - 0.03L \end{cases}$

Application (continued)

- $\ddot{B} + 0.09\dot{B} = 0$ $\quad L = \frac{100}{3}\dot{B} + 2B$

Initial Value Problem

- $\begin{cases} B(t) = c_1 + c_2 e^{-0.09t} \\ L(t) = 2c_1 - c_2 e^{-0.09t} \end{cases}$, with initial values $\begin{cases} B(0) = 3 \\ L(0) = 0 \end{cases}$