Matrix ODE Representations
Lecture 10b: 2023-03-20

MAT A35 – Winter 2023 – UTSC

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Homogeneous system as matrix equation

- We can rewrite a homogeneous system of 1st-order ODEs as a matrix-vector ODE.
- $\begin{cases} \dot{x} = ax + by \\ \dot{y} = cx + dy \end{cases} \quad \rightarrow \quad \underbrace{\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}}_{\dot{z}} = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{z}$

Ex. $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$\underbrace{\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} x + 0y \\ -4x + 2y \end{bmatrix}}$

- The matrix-form equation $\dot{z} = Az$ has solutions $z(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

Ex. $z = \begin{bmatrix} e^{t} \\ 4e^{t} - 2e^{2t} \end{bmatrix}$

$\dot{z} = \begin{bmatrix} e^{t} \\ 4e^{t} - 4e^{2t} \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} e^{t} \\ 4e^{t} - 2e^{2t} \end{bmatrix} = \begin{bmatrix} e^{t} \\ -4e^{t} + 8e^{t} - 4e^{2t} \end{bmatrix} = \begin{bmatrix} e^{t} \\ 4e^{t} - 4e^{2t} \end{bmatrix}$

Solution to a matrix-form equation

$y' - ky = 0 \quad (\lambda - k) = 0$

- If $y' = ky$, then $y = c_0 e^{kx}$ for single-variable ODEs.
- If $\dot{z} = Az$, then perhaps $\underline{z = e^{\lambda t} v}$, for some constant λ and vector v.

$$\dot{z} = \lambda e^{\lambda t} v$$

$$\lambda e^{\lambda t} v = A \left(e^{\lambda t} v \right)$$

$$e^{\lambda t} (\lambda v) = e^{\lambda t} (A v)$$

$$\lambda v = Av$$

- Notice that this is exactly the eigenvector/eigenvalue equation, so (λ, v) is an eigenpair of A.
- Thus, for each eigenpair (λ, v), $e^{\lambda t} v$ is a linearly independent solution. If we had n eigenpairs, we could get n solutions.

Using an eigenbasis for general solutions

- Let $\dot{z} = Az$, where A is an $n \times n$ matrix and $z(t)$ is a length-n vector.
- Then if $(\lambda_1, v_1), (\lambda_2, v_2), \ldots, (\lambda_n, v_n)$ is an eigenbasis of A, then
$$z(t) = c_1 v_1 e^{\lambda_1 t} + \cdots c_n v_n e^{\lambda_n t}$$
is the general solution for $z(t)$.

Ex. $A = \begin{bmatrix} 1 & 0 \\ -4 & 2 \end{bmatrix}$ Eigenvalues $\begin{vmatrix} \lambda-1 & 0 \\ 4 & \lambda-2 \end{vmatrix} = 0$

$(\lambda-1)(\lambda-2) = 0$

$\lambda = 1, 2$

$\lambda_1 = 1$, $\begin{bmatrix} 1 & 0 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \left.\begin{matrix} x = x \\ -4x + 2y = y \end{matrix}\right\} y = 4x$ $\quad v_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

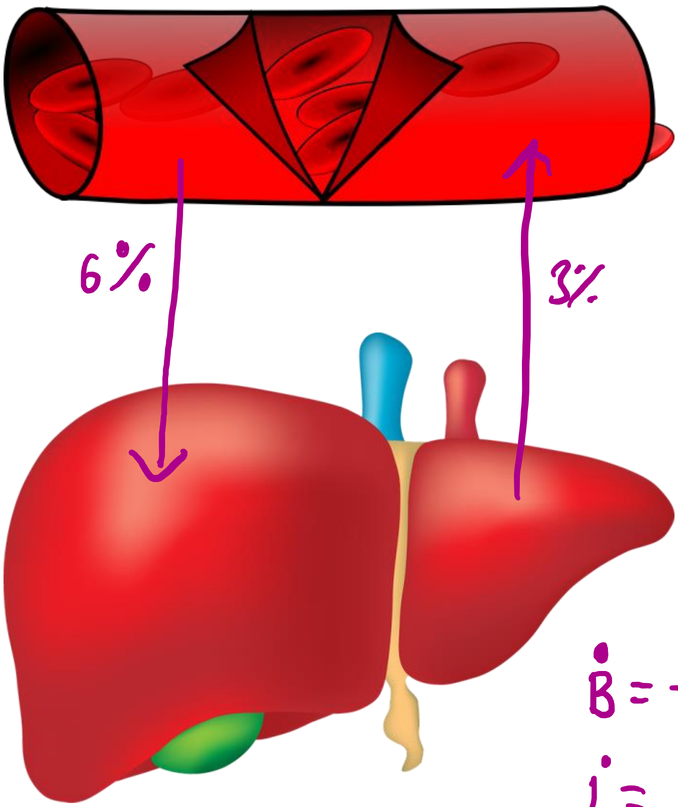
$\lambda_2 = 2$, $\begin{bmatrix} 1 & 0 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix} \Rightarrow \left.\begin{matrix} x = 2x \\ -4x + 2y = 2y \end{matrix}\right\} \begin{matrix} x = 0 \\ y = y \end{matrix}$ $\quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$z = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{2t}$

Ex. $\begin{bmatrix} e^{t} \\ 4e^{t} - 2e^{2t} \end{bmatrix}$

Application (Bittinger, 9.3, Ex. 5)

- 3mg of glactosyl human serum albumin (Tc-GSA) is injected into a patient's bloodstream to measure liver function. After injection, Tc-GSA is transferred from the blood to the liver at 6% per minute, and from the liver into blood at 3% per minute.
- How much Tc-GSA is in the blood or liver as a function of time?



$B(0) = 3$ mg

$L(0) = 0$ mg

$$B \underset{0.03L}{\overset{0.06B}{\rightleftarrows}} L$$

$$\dot{B} = -0.06B + 0.03L$$

$$\dot{L} = 0.06B - 0.03L$$

Application (continued)

- $\begin{cases} \dot{B} = -0.06B + 0.03L \\ \dot{L} = \ \ 0.06B - 0.03L \end{cases}$

$$0.03L = \dot{B} + 0.06B$$
$$3L = 100\dot{B} + 6B$$
$$L = \frac{100}{3}\dot{B} + 2B$$

$$\ddot{B} = -0.06\dot{B} + 0.03\dot{L}$$
$$\ddot{B} = -0.06\dot{B} + 0.03\left[0.06B - 0.03L\right]$$
$$\ddot{B} = -0.06\dot{B} + 0.0018B - 0.0009L$$
$$\ddot{B} = -0.06\dot{B} + 0.0018B - 0.0009\left[\frac{100}{3}\dot{B} + 2B\right]$$
$$\ddot{B} = -0.06\dot{B} + \underline{0.0018}B - 0.03\dot{B} \underline{- 0.0018B}$$
$$\ddot{B} + 0.09\dot{B} = 0$$

Application (continued)

- $\ddot{B} + 0.09\dot{B} = 0$ $\qquad L = \frac{100}{3}\dot{B} + 2B$

$\lambda^2 + 0.09\lambda = 0$

$\lambda(\lambda + 0.09) = 0$

$\lambda = 0, \ -0.09$

$B = c_1 e^{0t} + c_2 e^{-0.09t}$

$B = c_1 + c_2 e^{-0.09t}$

$\dot{B} = -0.09 c_2 e^{-0.09t}$

$L = -3c_2 e^{-0.09t} + 2c_1 + 2c_2 e^{-0.09t}$

$L = 2c_1 - c_2 e^{-0.09t}$

Initial Value Problem

- $\begin{cases} B(t) = c_1 + c_2 e^{-0.09t} \\ L(t) = 2c_1 - c_2 e^{-0.09t} \end{cases}$, with initial values $\begin{cases} B(0) = 3 \\ L(0) = 0 \end{cases}$

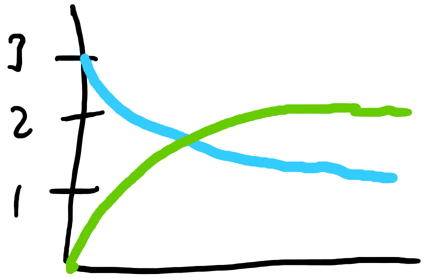
$$3 = B(0) = c_1 + c_2 e^{-0.09 \cdot 0} = c_1 + c_2$$

$$0 = L(0) = 2c_1 - c_2 e^{-0.09 \cdot 0} = 2c_1 - c_2$$

$$\Rightarrow \begin{matrix} 3 = 3c_1 \\ c_1 = 1 \end{matrix} \qquad \Rightarrow \quad c_2 = 2$$

$$B(t) = 1 + 2e^{-0.09t} \qquad L(t) = 2 - 2e^{-0.09t}$$

$\lim_{t\to\infty} B(t) = 1$ mg $\qquad \lim_{t\to\infty} L(t) = 2$ mg



Solving as a matrix equation

- $\begin{cases} \dot{B} = -0.06B + 0.03L \\ \dot{L} = \ \ 0.06B - 0.03L \end{cases}$ converts to $\begin{bmatrix} \dot{B} \\ \dot{L} \end{bmatrix} = \begin{bmatrix} -0.06 & 0.03 \\ 0.06 & -0.03 \end{bmatrix} \begin{bmatrix} B \\ L \end{bmatrix}$

$$|\lambda I - A| = 0$$

Eigendecomposition

$$0 = \begin{vmatrix} \lambda + 0.06 & -0.03 \\ -0.06 & \lambda + 0.03 \end{vmatrix} = \lambda^2 + 0.09\lambda + \underline{0.0018} - \underline{0.0018} = 0$$

$$\lambda^2 + 0.09\lambda = 0$$

$$\lambda(\lambda + 0.09) = 0 \Rightarrow \lambda = 0, -0.09$$

$$\lambda_1 = 0 \quad \begin{bmatrix} -0.06 & 0.03 \\ 0.06 & -0.03 \end{bmatrix} \begin{bmatrix} B \\ L \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} -0.06B + 0.03L = 0 \\ L = 2B \end{matrix} \quad v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\lambda_2 = -0.09 \quad \begin{bmatrix} -0.06 & 0.03 \\ 0.06 & -0.03 \end{bmatrix} \begin{bmatrix} B \\ L \end{bmatrix} = \begin{bmatrix} -0.09B \\ -0.09L \end{bmatrix} \Rightarrow \begin{matrix} -0.06B + 0.03L = -0.09B \\ 0.03L = -0.03B \\ L = -B \end{matrix} \quad v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Using eigenvectors for general solution

- $\lambda_1 = 0, v_1 = \begin{bmatrix}1\\2\end{bmatrix}$ and $\lambda_2 = -0.09, v_2 = \begin{bmatrix}1\\-1\end{bmatrix}$

$$\begin{bmatrix}B\\L\end{bmatrix} = c_1 e^{0t}\begin{bmatrix}1\\2\end{bmatrix} + c_2 e^{-0.09t}\begin{bmatrix}1\\-1\end{bmatrix}$$

$$\begin{bmatrix}B\\L\end{bmatrix} = \underline{c_1\begin{bmatrix}1\\2\end{bmatrix} + c_2 e^{-0.09t}\begin{bmatrix}1\\-1\end{bmatrix}} = \underline{\begin{bmatrix}c_1 + c_2 e^{-0.09t}\\2c_1 - c_2 e^{-0.09t}\end{bmatrix}}$$

Initial value problem can then be solved with $B(t)$ and $L(t)$

Try it out

$\dot{x} = 0x - 1y$

$\dot{y} = 3x + 2y$

$$\begin{bmatrix} 0 & -1 \\ 3 & 2 \end{bmatrix}$$

- $\begin{cases} \dot{x} = -y \\ \dot{y} = 3x + 2y \end{cases}$ and $x(0) = 0, y(0) = 4$

- Step 1. Convert to $z = Az$, where $z = \begin{bmatrix} x \\ y \end{bmatrix}$.

A: $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$

B: $A = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$

C: $A = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$

D: $A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}$

E: None of the above

- Step 2. Find the eigenvalues.

$$\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}$$

$\dot{x} = y$

$\dot{y} = 3x + 2$

$$\begin{vmatrix} \lambda & -1 \\ -3 & \lambda - 2 \end{vmatrix} = \lambda^2 - 2\lambda - 3 = 0$$

$$(\lambda - 3)(\lambda + 1) = 0$$

$$\lambda = -1, 3$$

A: $\lambda = -1, 1$

B: $\lambda = 1, 2$

C: $\lambda = 1, 3$

D: $\lambda = -1, 3$

E: None of the above

Try it out (continued)

- Find the eigenvectors.

A: $v_1 = \begin{bmatrix}1\\1\end{bmatrix}, v_2 = \begin{bmatrix}1\\3\end{bmatrix}$

B: $v_1 = \begin{bmatrix}1\\-1\end{bmatrix}, v_2 = \begin{bmatrix}1\\3\end{bmatrix}$

C: $v_1 = \begin{bmatrix}1\\1\end{bmatrix}, v_2 = \begin{bmatrix}1\\-3\end{bmatrix}$

D: $v_1 = \begin{bmatrix}1\\-1\end{bmatrix}, v_2 = \begin{bmatrix}1\\-3\end{bmatrix}$

E: None of the above

$$\begin{bmatrix}0 & 1\\3 & 2\end{bmatrix} \quad \lambda_1 = -1 \quad \lambda_2 = 3$$

$$\lambda_1 = -1 \quad \begin{bmatrix}0 & 1\\3 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}-x\\-y\end{bmatrix} \quad y = -x \quad v_1 = \begin{bmatrix}1\\-1\end{bmatrix}$$

$$\lambda_2 = 3 \quad \begin{bmatrix}0 & 1\\3 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}3x\\3y\end{bmatrix} \quad y = 3x \quad v_2 = \begin{bmatrix}1\\3\end{bmatrix}$$

- What's the general solution?

$$\begin{bmatrix}x\\y\end{bmatrix} = c_1 e^{-t}\begin{bmatrix}1\\-1\end{bmatrix} + c_2 e^{3t}\begin{bmatrix}1\\3\end{bmatrix}$$

A: $\begin{bmatrix}x\\y\end{bmatrix} = c_1 e^{-t}\begin{bmatrix}1\\1\end{bmatrix} + c_2 e^{3t}\begin{bmatrix}1\\3\end{bmatrix}$

B: $\begin{bmatrix}x\\y\end{bmatrix} = c_1 e^{-t}\begin{bmatrix}1\\-1\end{bmatrix}, + c_2 e^{3t}\begin{bmatrix}1\\3\end{bmatrix}$

C: $\begin{bmatrix}x\\y\end{bmatrix} = c_1 e^{-t}\begin{bmatrix}1\\1\end{bmatrix}, + c_2 e^{3t}\begin{bmatrix}1\\-3\end{bmatrix}$

D: $\begin{bmatrix}x\\y\end{bmatrix} = c_1 e^{-t}\begin{bmatrix}1\\-1\end{bmatrix}, + c_2 e^{3t}\begin{bmatrix}1\\-3\end{bmatrix}$

E: None of the above

Try it out (cont., Initial value problem)

- $x(\overset{t}{0}) = 0, y(\overset{t}{0}) = 4$

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 4 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ -c_1 + 3c_2 \end{bmatrix} \Rightarrow \begin{array}{r} c_1 + c_2 = 0 \\ -c_1 + 3c_2 = 4 \\ \hline 4c_2 = 4 \end{array} \Rightarrow \begin{array}{l} c_1 = -1 \\ c_2 = 1 \end{array}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = -e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + e^{3t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

A: $c_1 = -1, c_2 = -1$
B: $c_1 = 1, c_2 = -1$
$\rightarrow$ C: $c_1 = -1, c_2 = 1$
D: $c_1 = 1, c_2 = 1$
E: None of the above