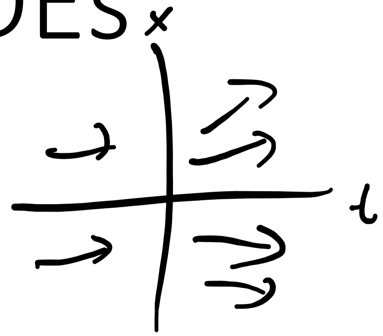
Phase portraits
Lecture 10c: 2023-03-23

MAT A35 – Winter 2023 – UTSC

Prof. Yun William Yu

System of two 1st-order ODEs



- $\begin{cases} \dot{x} = x + y - \sin t \\ \dot{y} = x^2 + y^2 - \ln t \end{cases}$ (nonautonomous)
- How many dimensions do nonautonomous systems need to draw direction fields?

← 3 for x, y, t

A: 1
B: 2
C: 3
D: 4
E: None of the above

- $\begin{cases} \dot{x} = x + y \\ \dot{y} = x^2 + y^2 \end{cases}$ (autonomous system)
- How many dimensions do autonomous systems need to draw direction fields?

← 3 for x, y, t

- How many dimensions do autonomous systems need to draw phase “lines”?

← 2 for just x, y

Plotting vector fields and trajectories

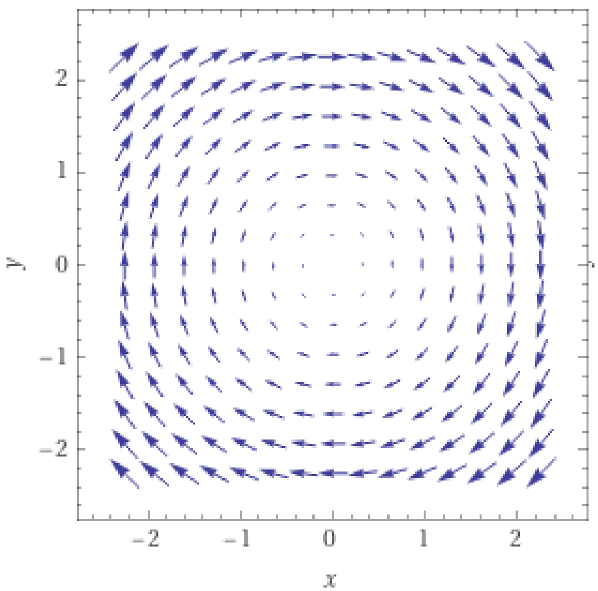
- Consider $\left\{\begin{aligned} \dot{x} &= f(x,y) \\ \dot{y} &= g(x,y) \end{aligned}\right\}$
- The system associates a direction and a magnitude for every point in $\mathbb{R}^2$, telling you what direction trajectories go.
- WolframAlpha: "vector field {f(x,y), g(x,y)}"
- Ex: "vector field {y, -x}" or "integral curves {y, -x}"
- Specify limits by adding "x=-3..3, y=-3..3" after

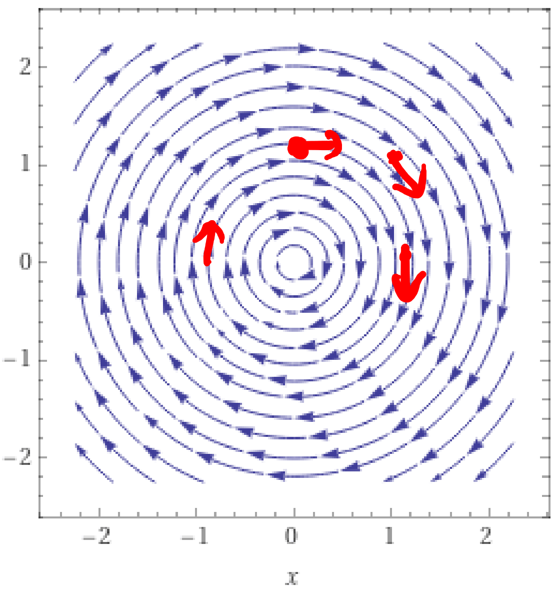
Ex. $\dot{x} = y$
$\dot{y} = -x$

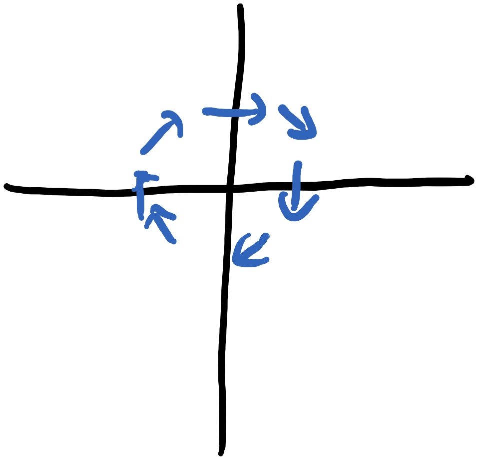
x, y	$\dot{x}$	$\dot{y}$
$(0,1)$	1	0
$(1,1)$	1	-1
$(1,0)$	0	-1
$(-1,0)$	0	1

vector field

trajectories







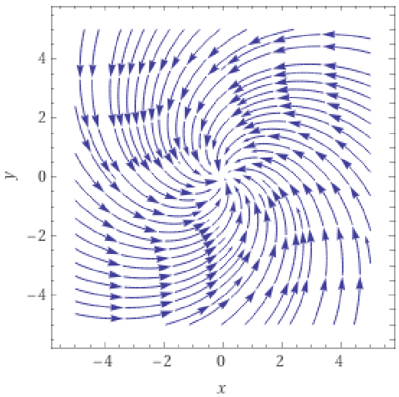
Try it out

- Which of the following is the integral curves for the system, plotted for x and y both between -5 and 5?

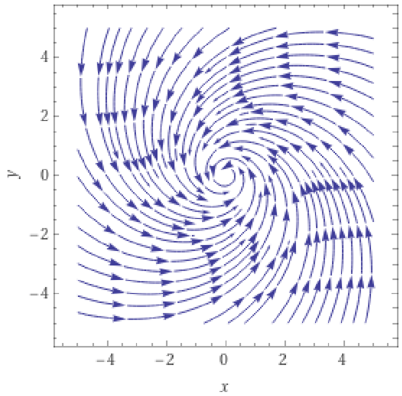
$$\dot{x} = 4x - y - \left(x + \frac{3}{2}y\right)(x^2 + y^2)$$
$$\dot{y} = x + 4y + \left(\frac{3}{2}x - y\right)(x^2 + y^2)$$

Wot polar

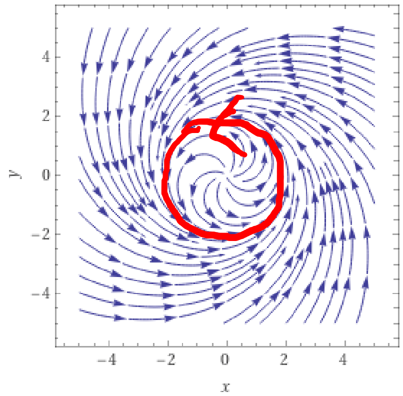
A:



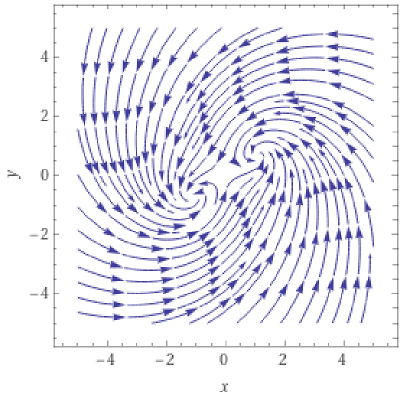
B:



C:



D:



https://www.wolframalpha.com/input/?i=integral+curves+%7B4*x-y-%28x%2B1.5y%29*%28x%5E2%2By%5E2%29%2C+x%2B4*y%2B%281.5x-y%29*%28x%5E2%2By%5E2%29%7D%2C+x%3D-5..5%2C+y%3D-5..5

Phase plane analysis

- Consider the autonomous homogeneous 2D linear system with constant coefficients

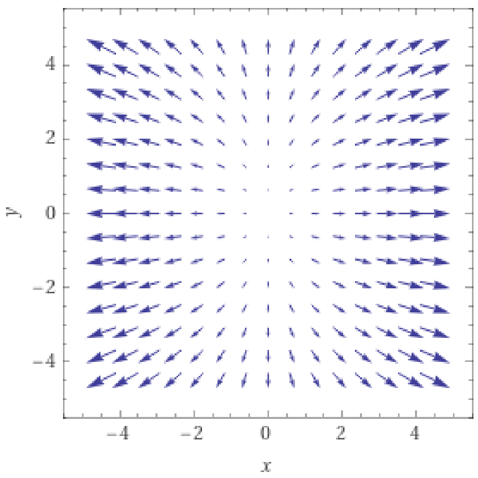
$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}, A = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$$

- Also, notice that the origin is always an equilibrium for a linear system.

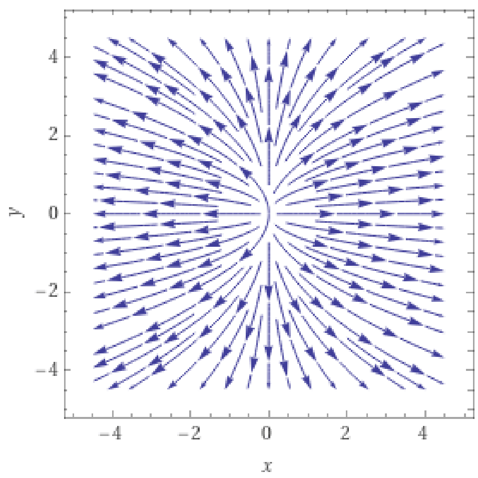
Ex. $A = \begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}$

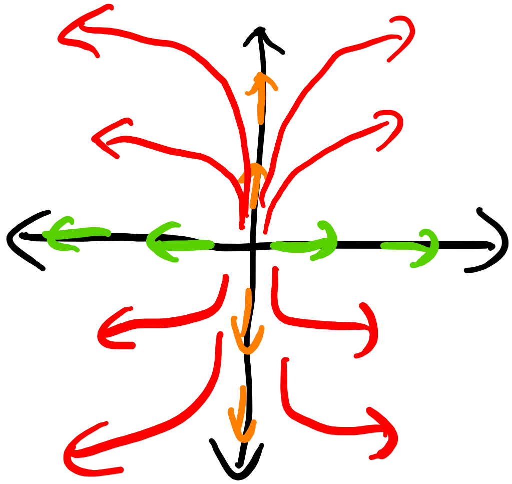
$$\begin{bmatrix}x\\ y\end{bmatrix} = \underline{c_1 e^{2t}\begin{bmatrix}1\\ 0\end{bmatrix}} + \underline{c_2 e^{t}\begin{bmatrix}0\\ 1\end{bmatrix}}$$

vector field



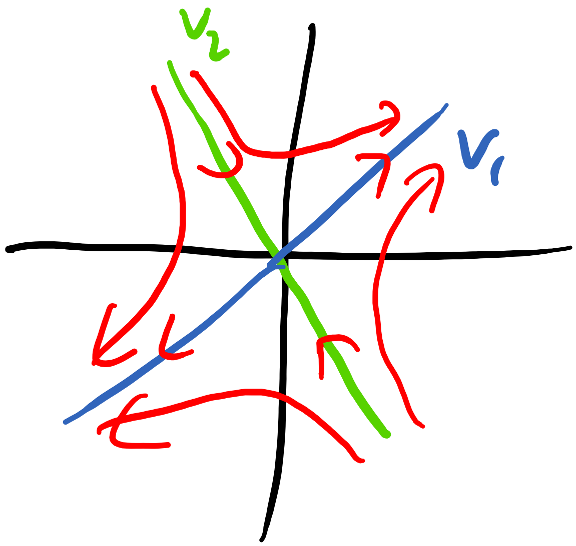
trajectories

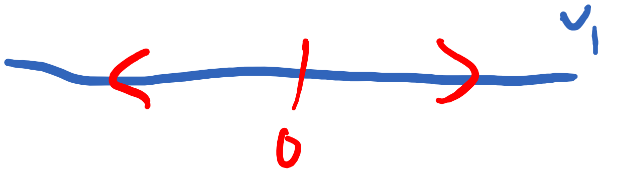


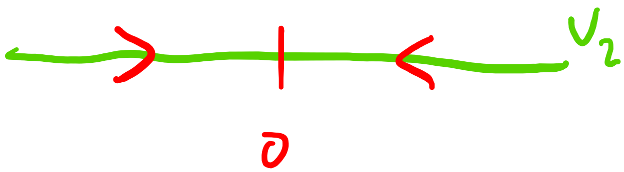


Using the eigendecomposition

- If (λ_1, v_1) and (λ_2, v_2) is an eigendecomposition of A, then the general solution describing any trajectory is
$$c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$$
- We can qualitatively analyze the behavior of the system by looking at the eigenvalues and eigenvectors.
- Consider $f(t) = ce^{\lambda t}$.
- If $\lambda > 0$, we get exponential growth away from 0.
- If $\lambda < 0$, we get exponential decay towards 0.







Sign and stability

- If eigenvalues are positive (or have a positive real part), then trajectories go away from the origin. (unstable node)
- If eigenvalues are negative (or have a negative real part), then trajectories go towards the origin. (asymptotically stable node)
- If eigenvalues have opposite signs, then we have a saddle point, as trajectories come in along one eigenvector, and leave along the other. (unstable, saddle point)

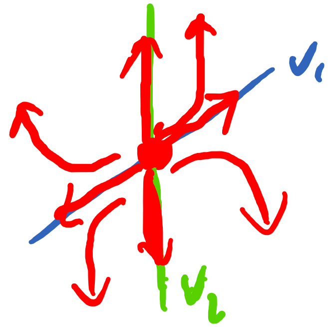
Ex. $\lambda = 2, 5$

$z = c_1 e^{2t} v_1 + c_2 e^{5t} v_2$

At $t = \infty$, both terms get bigger

runs away faster

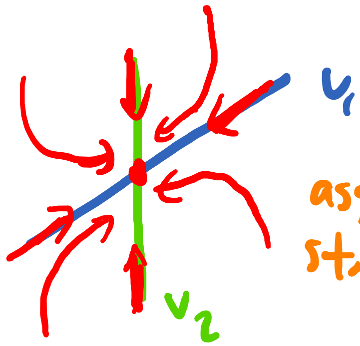
unstable



Ex. $\lambda = -2, -5$

$z = c_1 e^{-2t} v_1 + c_2 e^{-5t} v_2$

At $t = \infty$, both go to 0.



asymp. stable

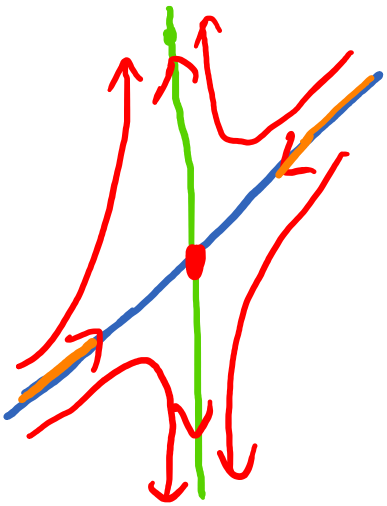
Ex. $\lambda = -2, 5$

$z = c_1 e^{-2t} v_1 + c_2 e^{5t} v_2$

$\begin{bmatrix}1\\1\end{bmatrix}$ $\begin{bmatrix}6\\-1\end{bmatrix}$

saddle pt.

unstable



Complex eigenvalues

- Recall complex eigenvalues come in pairs $\lambda_{1,2} = a \pm bi$.
- Solutions look like

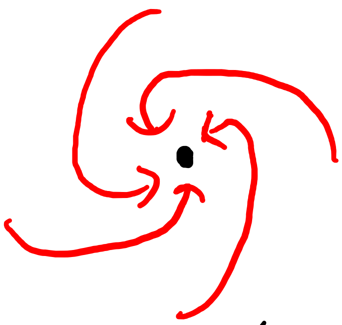
rotation

$$z = c_1 v_1 e^{at} \cos bt + c_2 v_2 e^{at} \sin bt$$

stability

- The sign of the real part a determines if the trajectories go inward (stable) or outward (unstable).
- The imaginary term means that the trajectories have a rotational component; i.e. might spiral in or out, or form a circle.

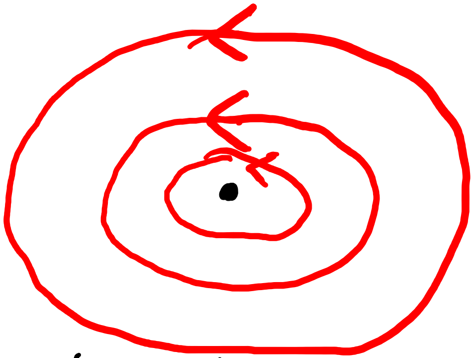
If $a < 0$



asymp. stable

spiral

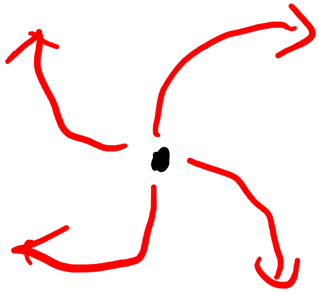
If $a = 0$



"stable"

center

If $a > 0$



unstable

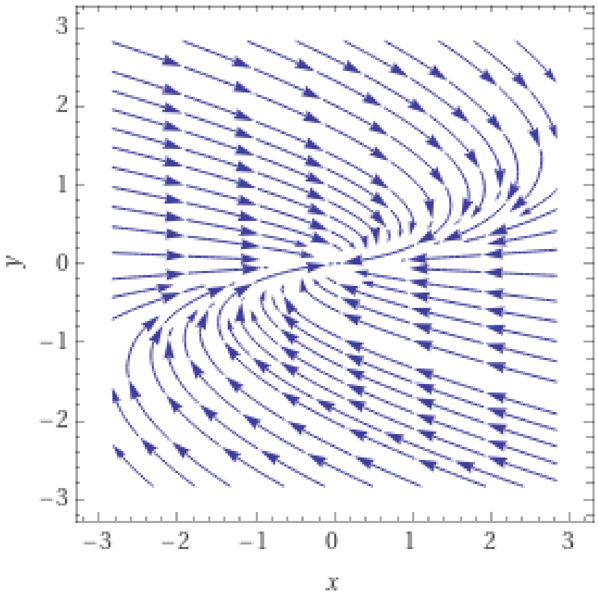
spiral

(Im)proper nodes

$A = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}$

- Sometimes, if $\lambda_1 = \lambda_2$, there is only one eigenvector. Then we have an *improper* node that's hard to draw.
    - Sign still determines stable vs unstable.

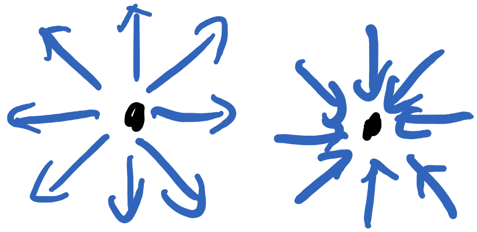
asymp. stable



- If $\lambda_1 = \lambda_2$ and we have two eigenvectors, then we have a *proper* node, which looks like a star.

Note: $A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

for proper nodes



Summarizing everything

- $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
- The origin (0,0) is always an equilibrium point.
- We can understand the behavior around the origin by looking at the eigenvalues of A.
- Positive real parts mean that the trajectories go outward.
- Negative real parts mean that the trajectories go inward.
- Opposite sign eigenvalues mean you have a saddle point.
- Nonzero imaginary components mean that trajectories spiral.

Try it out

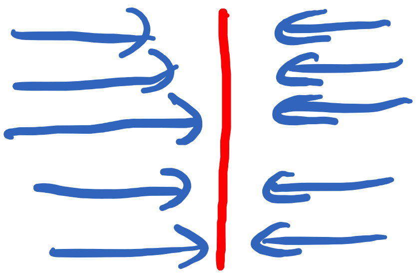
- $\lambda_1 = 4, \lambda_2 = -2$ unstable saddle pt
- $\lambda_1 = -3, \lambda_2 = -1$ asymp. stable node
- $\lambda_1 = 2, \lambda_2 = 3$ unstable node
- $\lambda_1 = 3, \lambda_2 = 3$ unstable node
- $\lambda_1 = 3 + 2i, \lambda_2 = 3 - 2i$ unstable spiral
- $\lambda_{1,2} = -1 \pm 2i$ asymp. stable spiral
- $\lambda_{1,2} = 0 \pm 4i$ stable center

A: Asymptotically Stable
B: Stable
C: Unstable
D: ???
E: None of the above

Special note: weird stuff can happen when $\lambda = 0$, which we won't deal with.

$\lambda = 0, -1$

$0 + 0i$



A: Node (incl. (im)proper)
B: Saddle Point
C: Spiral
D: Center
E: None of the above

Example

- Classify the behavior around the origin of

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}1 & 3\\ 3 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$$

Eig:

$$\begin{vmatrix}\lambda-1 & -3\\ -3 & \lambda-1\end{vmatrix} = (\lambda-1)^2 - 9 = 0$$

$$\lambda^2 - 2\lambda - 8 = 0$$

$$(\lambda-4)(\lambda+2) = 0$$

$$\lambda = 4, -2$$

Unstable Saddle pt.

Example

- Classify the behavior around the origin of

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$$

E.g.

$$\begin{vmatrix}\lambda-1 & -3\\ 0 & \lambda-1\end{vmatrix} = (\lambda-1)^2 = 0$$

$\lambda = 1$, multiplicity 2.

(improper)
Unstable
Node

$$\begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}x\\ y\end{bmatrix}$$

$\Rightarrow x + 3y = x$

$\Rightarrow y = 0$

$\Rightarrow v_1 = \begin{bmatrix}1\\ 0\end{bmatrix}$

Try it out

- Classify the behavior around the origin of

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

<u>Eig.</u> $0 = \begin{vmatrix} \lambda-1 & 3 \\ -3 & \lambda-1 \end{vmatrix} = (\lambda-1)^2 + 9 = 0$

$(\lambda-1)^2 = -9$

$\lambda - 1 = \pm 3i$

$\lambda = 1 \pm 3i$

unstable spiral

A: Asymptotically Stable
B: Stable
C: Unstable
D: ???
E: None of the above

A: Node (incl. (im)proper)
B: Saddle Point
C: Spiral
D: Center
E: None of the above