Nonlinear phase portraits
Lecture 11a: 2023-03-27

MAT A35 – Winter 2023 – UTSC

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Summarizing linear phase portraits

- $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
- The origin (0,0) is always an equilibrium point.
- We can understand the behavior around the origin by looking at the eigenvalues of A.
- Positive real parts mean that the trajectories go outward.
- Negative real parts mean that the trajectories go inward.
- Opposite sign eigenvalues mean you have a saddle point.
- Nonzero imaginary components mean that trajectories spiral.

Nonlinear autonomous systems and Jacobians

- $\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases}$
- Equilibrium points when $\begin{cases} \dot{x} = 0 \\ \dot{y} = 0 \end{cases}$
- We can approximate a function around a point using its derivative at that point.
- The *Jacobian* of the system is the analogue of the derivative:

$$J(x,y) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$$

Ex.

$$\dot{x} = \underline{x^2+y^2-1}$$
$$\dot{y} = \underline{2xy}$$

$\rightarrow x^2+y^2-1=0$

$2xy=0$

$\Rightarrow x=0$ or $y=0$

If $x=0$, $y^2-1=0 \Rightarrow y=\pm 1$

If $y=0$, $x^2-1=0 \Rightarrow x=\pm 1$

Equilibria: $\underline{(0,1)}$, $\underline{(0,-1)}$, $\underline{(1,0)}$, $\underline{(-1,0)}$

$$J(x,y) = \begin{bmatrix} 2x & 2y \\ 2y & 2x \end{bmatrix}$$

$$J(0,1) = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$$

Nonlinear equilibria behavior

$Av = \lambda v$

- Around each equilibrium, we can approximate its behavior by looking at the Jacobian matrix' eigenvalues.

A: $|\lambda I - A| = 0$

B: $|A - \lambda I| = 0$

C: Both

$$J(x,y) = \begin{bmatrix} 2x & 2y \\ 2y & 2x \end{bmatrix}$$

$$J(0,1) = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \qquad \begin{vmatrix} \lambda & -2 \\ -2 & \lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 4 = 0, \quad \lambda = \pm 2$$

saddle pt

$\lambda_1 = 2$, $\lambda_2 = -2$

$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$J(0,-1) = \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix} \qquad \begin{vmatrix} \lambda & 2 \\ 2 & \lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 4 = 0, \quad \lambda = \pm 2$$

saddle pt

$\lambda_1 = 2$, $\lambda_2 = -2$

$v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$J(1,0) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad \lambda = 2, \text{ mult } 2.$$

unstable node

$\cancel{v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$ ~~(proper)~~

$$J(-1,0) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \qquad \lambda = -2, \text{ mult } 2$$

stable node

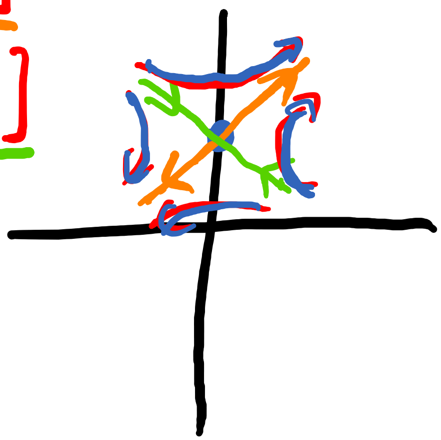
Local phase portraits

$(0, 1)$

eigenvectors

$\lambda_1 = 2 \quad v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

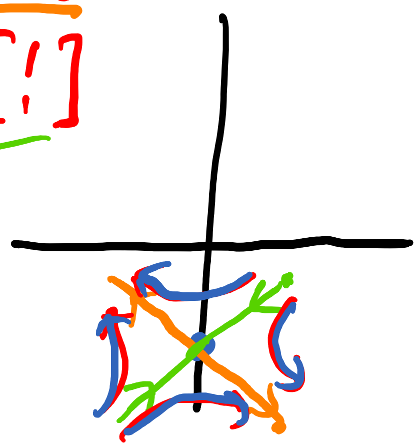
$\lambda_2 = -2 \quad v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$



$(0, -1)$

$\lambda_1 = 2 \quad v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

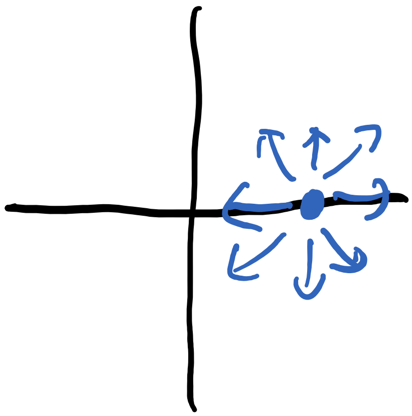
$\lambda_1 = -2 \quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



$(1, 0)$

$\lambda = 2$, mult 2

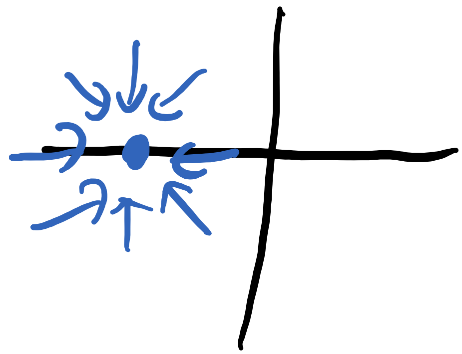
node



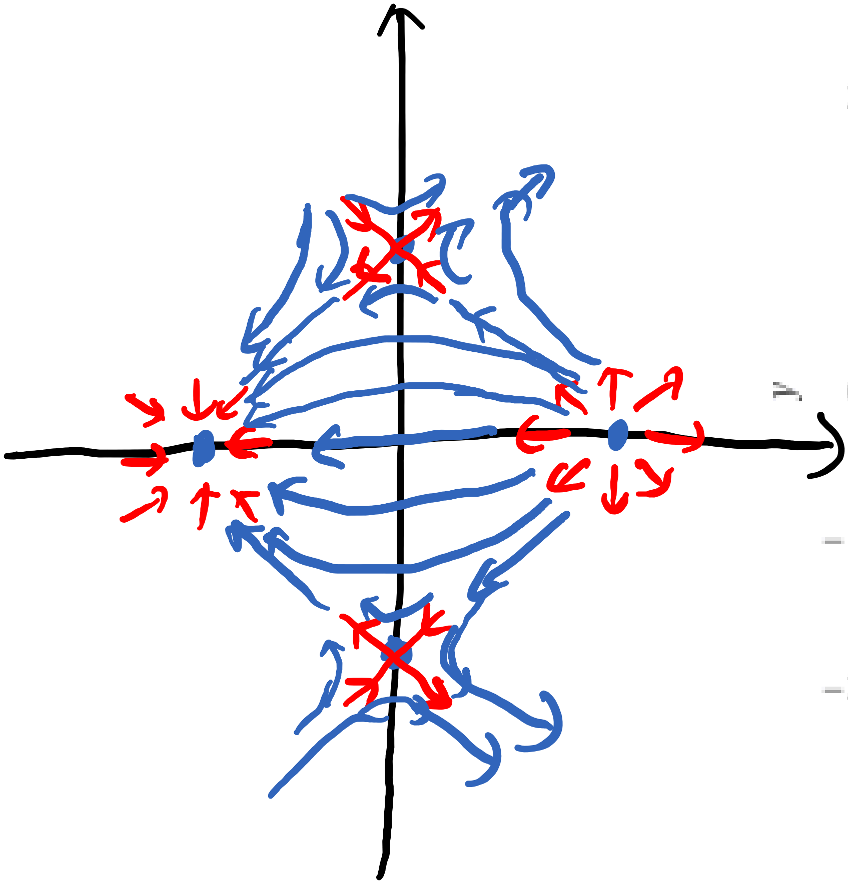
$(-1, 0)$

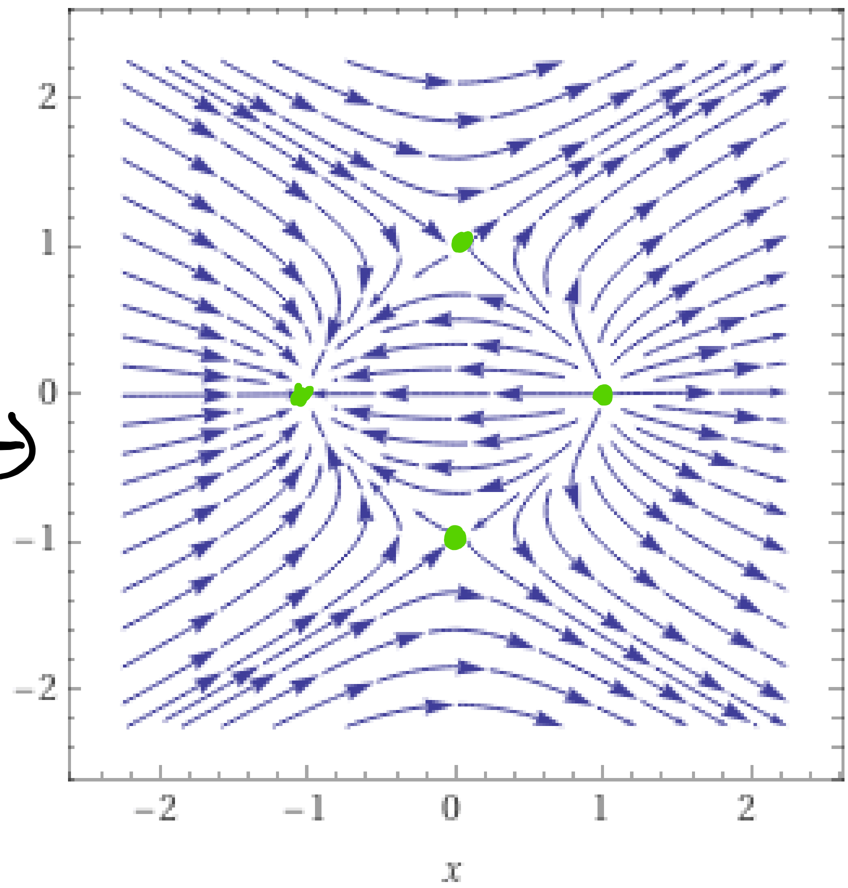
$\lambda = -2$ mult 2

node



Phase portraits





Try it out

- Find the equilibrium points for the nonlinear system:

$$0 = \dot{x} = xy$$
$$0 = \dot{y} = x + 2y - 8$$

$xy = 0$, $\quad x + 2y - 8 = 0$

Case 1: $x = 0$

$2y - 8 = 0 \Rightarrow y = 4$ $\qquad (0, 4)$

Case 2: $y = 0$

$x - 8 = 0 \Rightarrow x = 8$ $\qquad (8, 0)$

A: (0, 4) and (8, 0)
B: (4,8) and (0,0)
C: (-4, 0) and (-8, 0)
D: (-4, -8) and (4, 8)
E: None of the above

Try it out

- Find the Jacobian matrix for the nonlinear system:

$$\dot{x} = xy$$
$$\dot{y} = x + 2y - 8$$

A: $\begin{bmatrix} x & y \\ x & 2y \end{bmatrix}$

B: $\begin{bmatrix} x & y \\ 1 & 2 \end{bmatrix}$

C: $\begin{bmatrix} x & y \\ 2y & x \end{bmatrix}$

D: $\begin{bmatrix} y & x \\ 1 & 2 \end{bmatrix}$

E: None of the above

$$J(x,y) = \begin{bmatrix} \frac{\partial}{\partial x}[xy] & \frac{\partial}{\partial y}[xy] \\ \frac{\partial}{\partial x}[x+2y-8] & \frac{\partial}{\partial y}[x+2y-8] \end{bmatrix}$$

$$= \begin{bmatrix} y & x \\ 1 & 2 \end{bmatrix}$$

Try it out

- Use the Jacobian to determine the behavior around the first equilibrium point: $(0, 4)$

$$J(x,y) = \begin{bmatrix} y & x \\ 1 & 2 \end{bmatrix}$$

$$J(0,4) = \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix}$$

$$\begin{vmatrix} 4-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

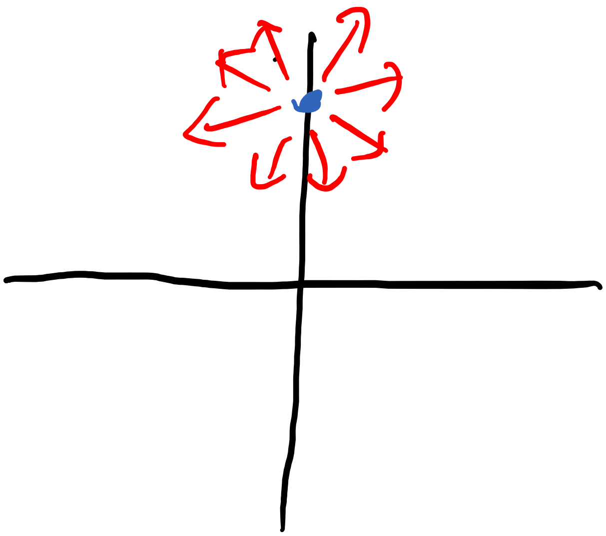
$$(4-\lambda)(2-\lambda) = 0$$

$$\lambda = 4, 2$$

unstable node

A: Asymptotically stable
B: Stable
C: Unstable
D: ???
E: None of the above

A: Node
B: Saddle point
C: Spiral
D: Center
E: None of the above



Try it out

- Use the Jacobian to determine the behavior around the second equilibrium point: $(8,0)$

A: Asymptotically stable
B: Stable
C: Unstable
D: ???
E: None of the above

A: Node
B: Saddle point
C: Spiral
D: Center
E: None of the above

$$J(x,y) = \begin{bmatrix} y & x \\ 1 & 2 \end{bmatrix}$$

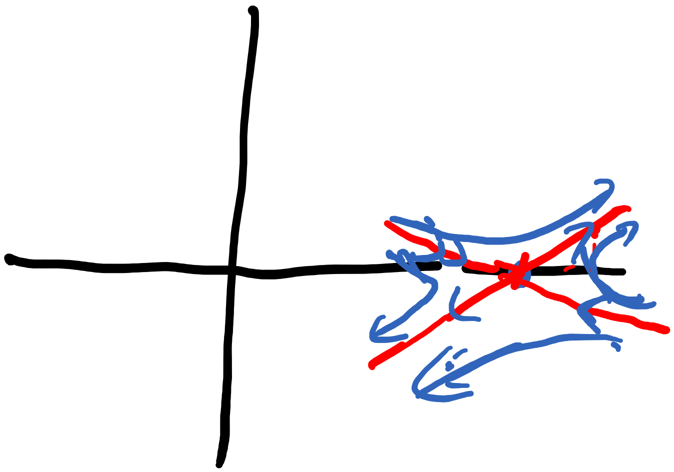
$$J(8,0) = \begin{bmatrix} 0 & 8 \\ 1 & 2 \end{bmatrix}$$

$$\begin{vmatrix} \lambda & -8 \\ -1 & \lambda-2 \end{vmatrix} = 0$$

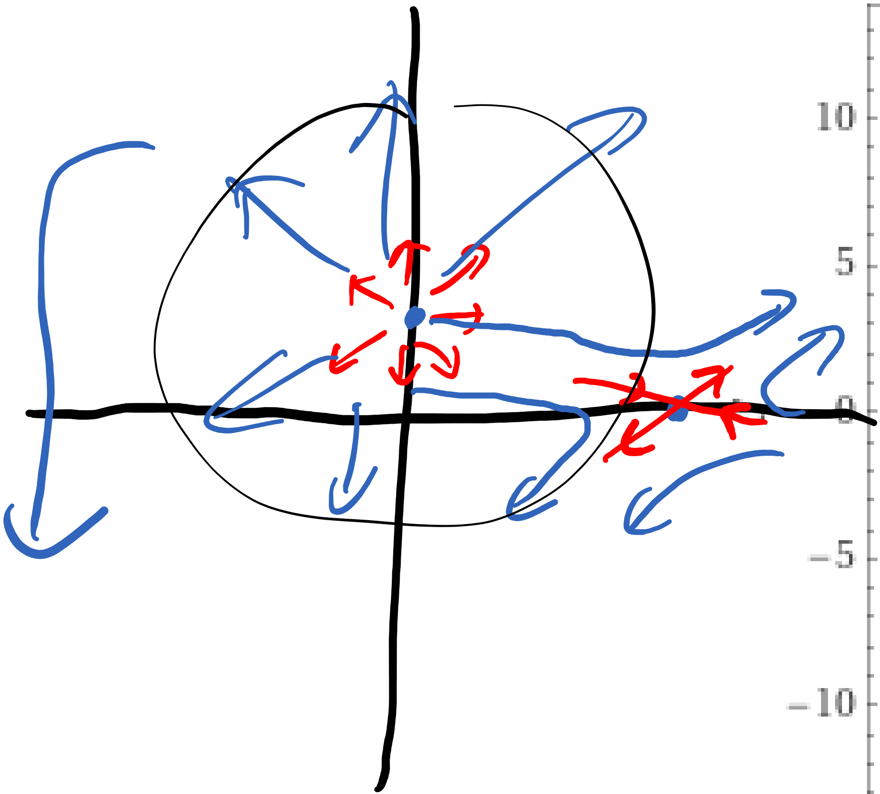
$$\lambda^2 - 2\lambda - 8 = 0$$
$$(\lambda+2)(\lambda-4) = 0$$
$$\lambda = -2,\ 4$$

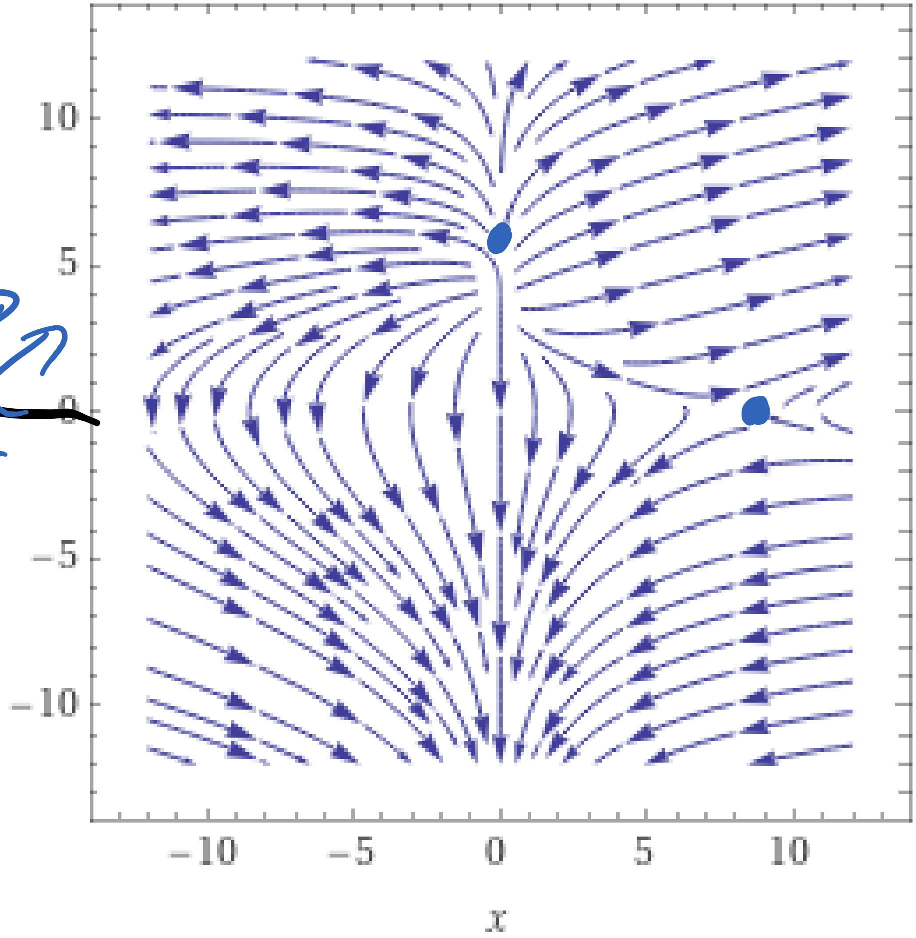
$\lambda_1 = -2 \qquad 8y = -2x \qquad v_1 = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$

$\lambda_2 = 4 \qquad 8y = 4x \qquad v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$



Putting it together





Predator-prey Lotka-Volterra model

- Consider an environment with wolves and deer.
- Deer population: $x(t)$
- Wolf population: $y(t)$
- The deer grow exponentially, but the population is kept in check by predation from the deer:

$$\dot{x}(t) = \underline{2x} - \underline{xy}$$

interactions are bad for deer

- The wolves die out, unless they can find enough deer to eat.

$$\dot{y}(t) = \underline{-y} + \underline{0.4xy}$$

interactions good for wolves

https://commons.wikimedia.org/wiki/File:Wolves_eating_deer.jpeg

Find equilibrium values and stability

- $\begin{cases} \dot{x} = 2x - xy \\ \dot{y} = -y + 0.4xy \end{cases}$

$$J(x,y) = \begin{bmatrix} 2-y & -x \\ 0.4y & -1+0.4x \end{bmatrix}$$

$0 = 2x - xy = x(2-y) \Rightarrow x = 0$ or $y = 2$

$0 = -y + 0.4xy = y(0.4x - 1)$

Case 1: $x = 0 \Rightarrow y = 0$ $(0,0)$

Case 2: $y = 2 \Rightarrow 0 = 0.4x - 1 \Rightarrow x = 2.5$ $(2.5, 2)$

$$J(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \left.\begin{matrix} \lambda_1 = 2 & v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \lambda_2 = -1 & v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{matrix}\right\} \text{saddle pt}$$

$$J(2.5,2) = \begin{bmatrix} 0 & -2.5 \\ 0.8 & 0 \end{bmatrix} \Rightarrow \left.\begin{matrix} \lambda^2 + 0.25 \cdot 0.8 = 0 \\ \lambda^2 + 2 = 0 \\ \lambda = \pm i\sqrt{2} \end{matrix}\right\} \text{center}$$

Phase diagram