# Nonlinear phase portraits
# Lecture 11a: 2023-03-27

MAT A35 – Winter 2023 – UTSC

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# Summarizing linear phase portraits

- $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
- The origin (0,0) is always an equilibrium point.
- We can understand the behavior around the origin by looking at the eigenvalues of $A$.
- Positive real parts mean that the trajectories go outward.
- Negative real parts mean that the trajectories go inward.
- Opposite sign eigenvalues mean you have a saddle point.
- Nonzero imaginary components mean that trajectories spiral.

# Nonlinear autonomous systems and Jacobians

- $\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = g(x, y) \end{cases}$
- Equilibrium points when $\begin{cases} \dot{x} = 0 \\ \dot{y} = 0 \end{cases}$
- We can approximate a function around a point using its derivative at that point.
- The *Jacobian* of the system is the analogue of the derivative:

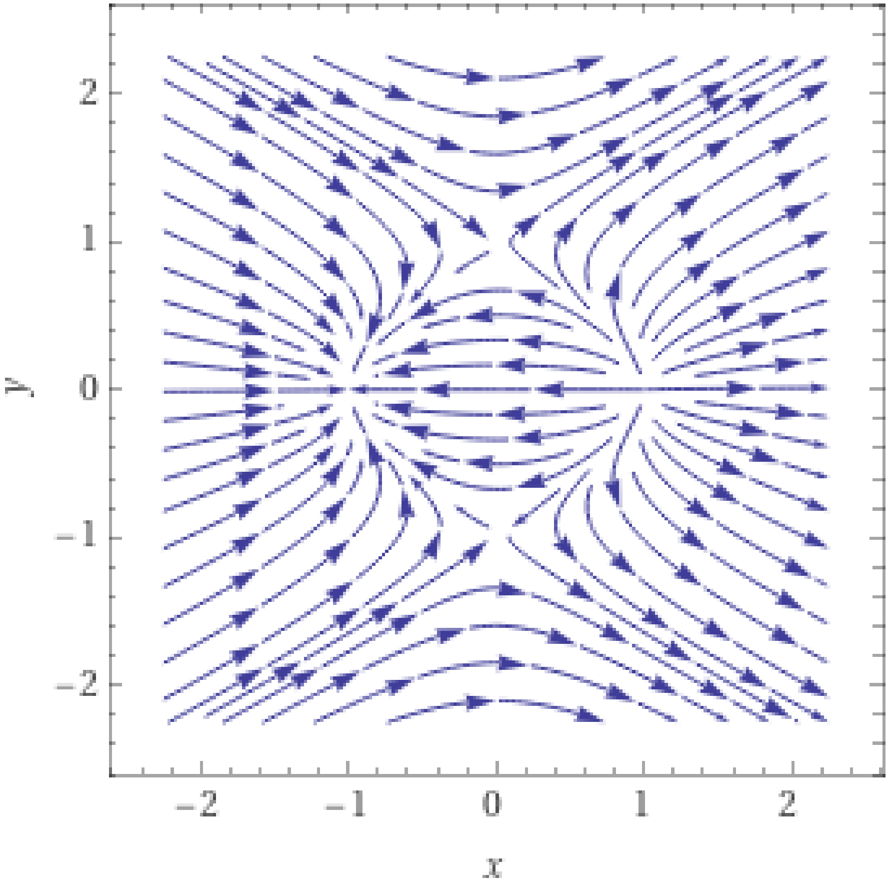
$$J(x, y) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$$

# Nonlinear equilibria behavior

- Around each equilibrium, we can approximate its behavior by looking at the Jacobian matrix' eigenvalues.

# Local phase portraits

# Phase portraits



# Try it out

- Find the equilibrium points for the nonlinear system:

$$\dot{x} = xy$$
$$\dot{y} = x + 2y - 8$$

A: (0, 4) and (8, 0)
B: (4,8) and (0,0)
C: (-4, 0) and (-8, 0)
D: (-4, -8) and (4, 8)
E: None of the above

# Try it out

- Find the Jacobian matrix for the nonlinear system:

$$\dot{x} = xy$$
$$\dot{y} = x + 2y - 8$$

A: $\begin{bmatrix} x & y \\ x & 2y \end{bmatrix}$

B: $\begin{bmatrix} x & y \\ 1 & 2 \end{bmatrix}$

C: $\begin{bmatrix} x & y \\ 2y & x \end{bmatrix}$

D: $\begin{bmatrix} y & x \\ 1 & 2 \end{bmatrix}$

E: None of the above

# Try it out

- Use the Jacobian to determine the behavior around the first equilibrium point:

A: Asymptotically stable
B: Stable
C: Unstable
D: ???
E: None of the above

A: Node
B: Saddle point
C: Spiral
D: Center
E: None of the above

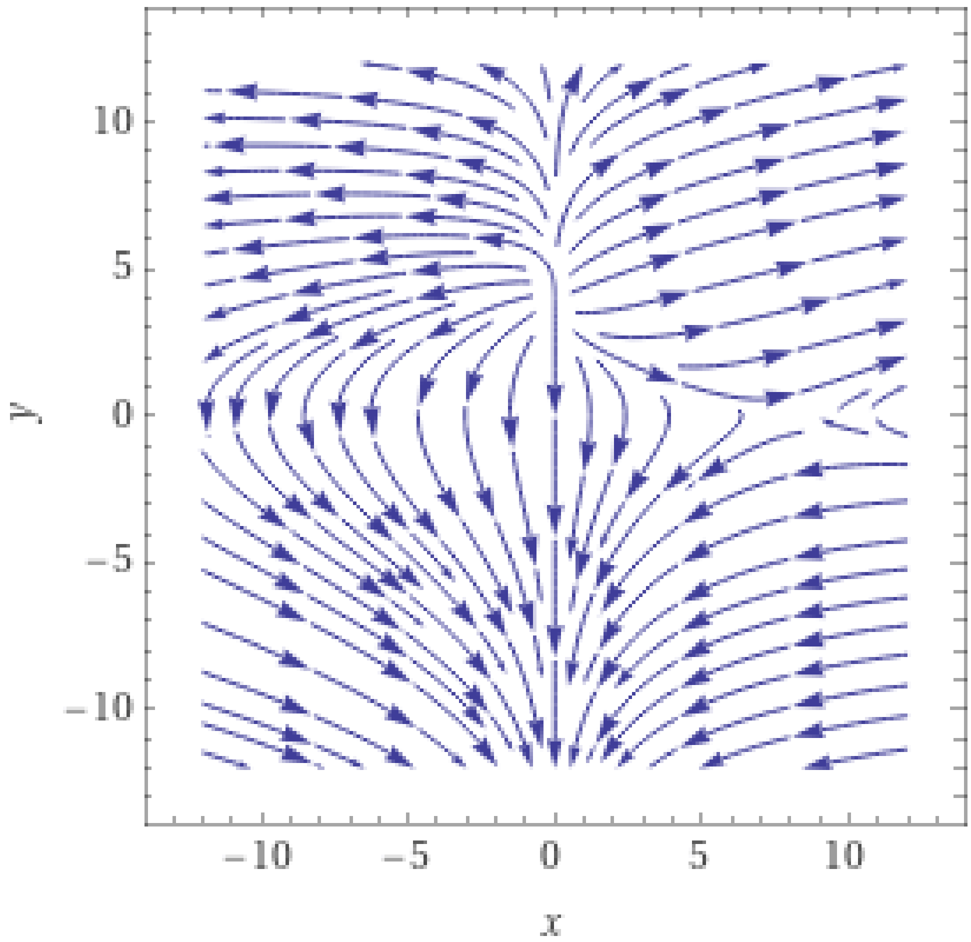
# Try it out

- Use the Jacobian to determine the behavior around the second equilibrium point:

A: Asymptotically stable
B: Stable
C: Unstable
D: ???
E: None of the above

A: Node
B: Saddle point
C: Spiral
D: Center
E: None of the above

# Putting it together



# Predator-prey Lotka-Volterra model

- Consider an environment with wolves and deer.
- Deer population: $x(t)$
- Wolf population: $y(t)$
- The deer grow exponentially, but the population is kept in check by predation from the deer:
  $$\dot{x}(t) = 2x - xy$$
- The wolves die out, unless they can find enough deer to eat.
  $$\dot{y}(t) = -y + 0.4xy$$

https://commons.wikimedia.org/wiki/File:Wolves_eating_deer.jpeg

# Find equilibrium values and stability

- $\begin{cases} \dot{x} = 2x - xy \\ \dot{y} = -y + 0.4xy \end{cases}$

# Phase diagram