Power series
Lecture 11b: 2023-03-27

MAT A35 – Winter 2023 – UTSC

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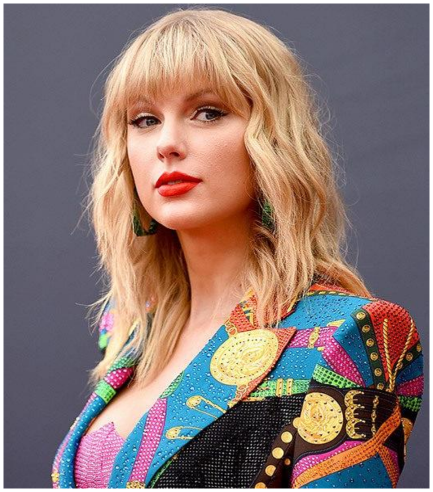
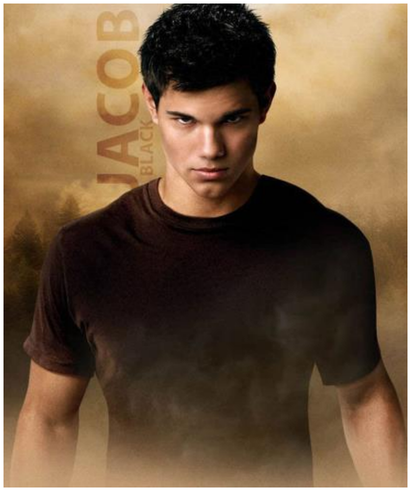
Who is your favorite Taylor?

A: Taylor Swift



B: Taylor Lautner



C: Brook Taylor

D: Taylor Gun-Jin Wang

E: Taylor Momsen

Simple mathematical operations

- Which mathematical operation is the hardest?
- Adding, subtracting, or multiplying two real numbers gives a real number.

A: Addition
B: Subtraction
C: Multiplication
D: Division
E: All are equally hard

$5+3=8$ $\qquad$ $4\cdot 0=0$

$3-5=-2$ $\qquad$ $\pi\cdot e$ is real

- Dividing two real numbers may not.

$4/5=0.8$

$4/0$ is not a number

Polynomials

- A real polynomial $p(x)$ in a variable x is an expression that combines together x with real numbers using just addition, subtraction, and multiplication, but no division.

poly: $p(x) = (x+1)(x-1) + 5x \cdot x - 3.4x + \pi$

not poly: $f(x) = \frac{1}{x-1}$ $f(x) = e^x$ $f(x) = (\ln x) x^2$

- Canonical form for nth-order polynomials:

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_i \in \mathbb{R}$ and n is a positive integer.

Ex. $p(x) = x^2 - 1 + 5x^2 - 3.4x + \pi$

$= 6x^2 + (-3.4)x + (\pi - 1)$

Polynomials are nice

- Polynomials are built up from the "easy" operations of addition and multiplication (and implicitly subtraction).
- If you add together two polynomials, you get another polynomial.

$p(x) = x^2+2x+1 \qquad q(x) = x^3-1 \qquad p(x)+q(x) = x^3+x^2+2x$

- If you multiply together two polynomials, you get another polynomial.

$p(x)q(x) = x^5+2x^4+x^3-x^2-2x-1$

- Polynomials are infinitely "smooth" meaning you can keep on taking derivatives at any point.

$p'(x) = 2x+2 \qquad p'''(x) = 0$

$p''(x) = 2 \qquad p''''(x) = 0$

$\vdots$

non example

$f(x) = |x|$

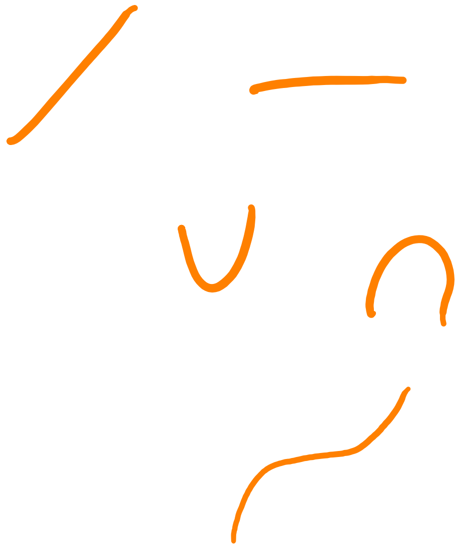
$f'(x) = \begin{cases} -1, & x<0 \\ 1, & x>0 \\ \text{undefined}, & x=0 \end{cases}$

Recall: different types of regression

- Linear regression:
$$f(x) = mx + b$$
- Quadratic regression:
$$f(x) = m_2x^2 + m_1x + b$$
- Cubic regression:
$$f(x) = m_3x^3 + m_2x^2 + m_1x$$
- Polynomial regression of degree n:
$$f(x) = b + \sum_{i=1}^{n} m_i x^i$$

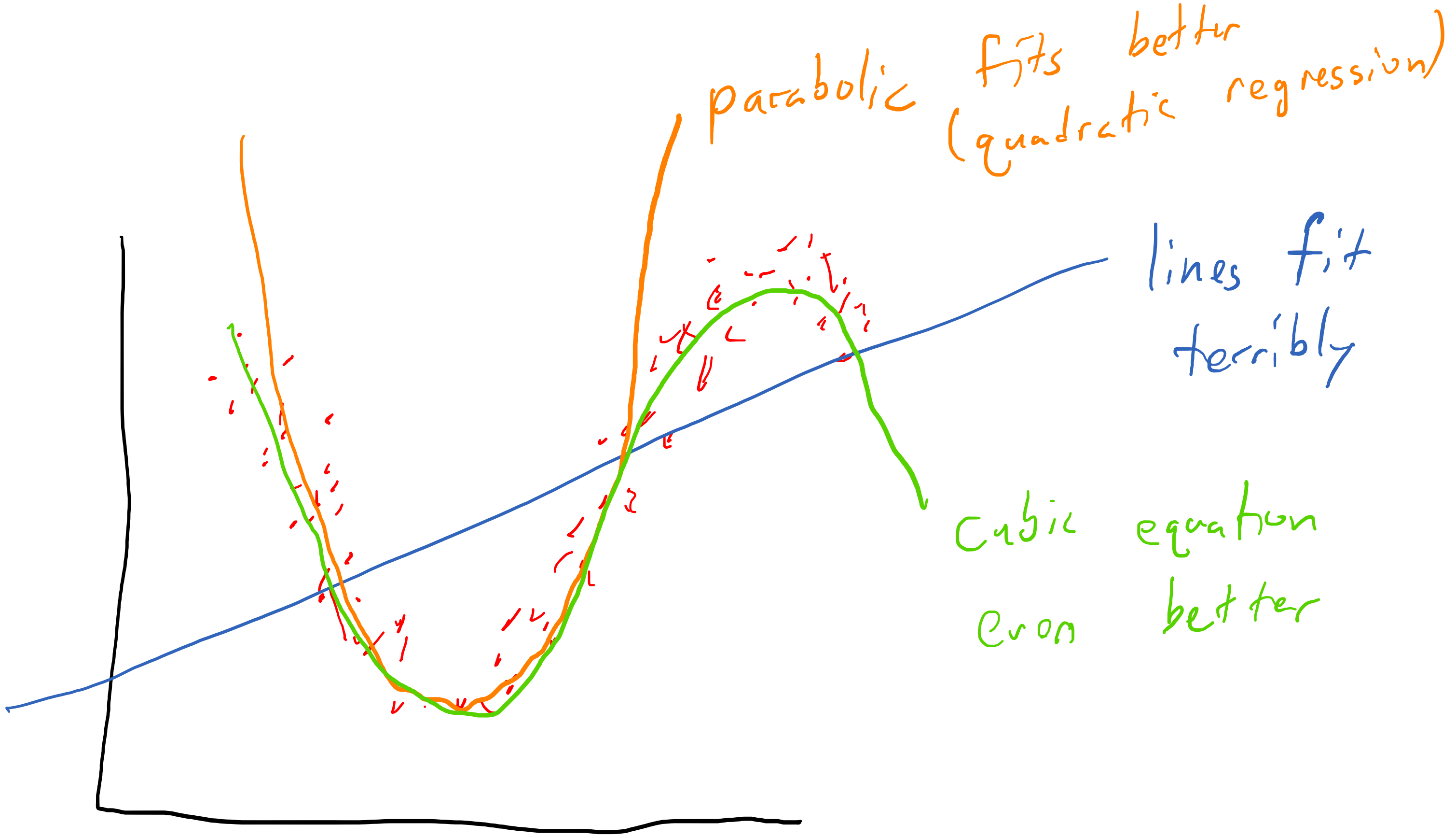
can add more parameters for better fit.

- Exponential regression:
$$f(x) = c_1 e^{c_2 x}$$
- Power dependencies:
$$f(x) = c_1 x^{c_2}$$



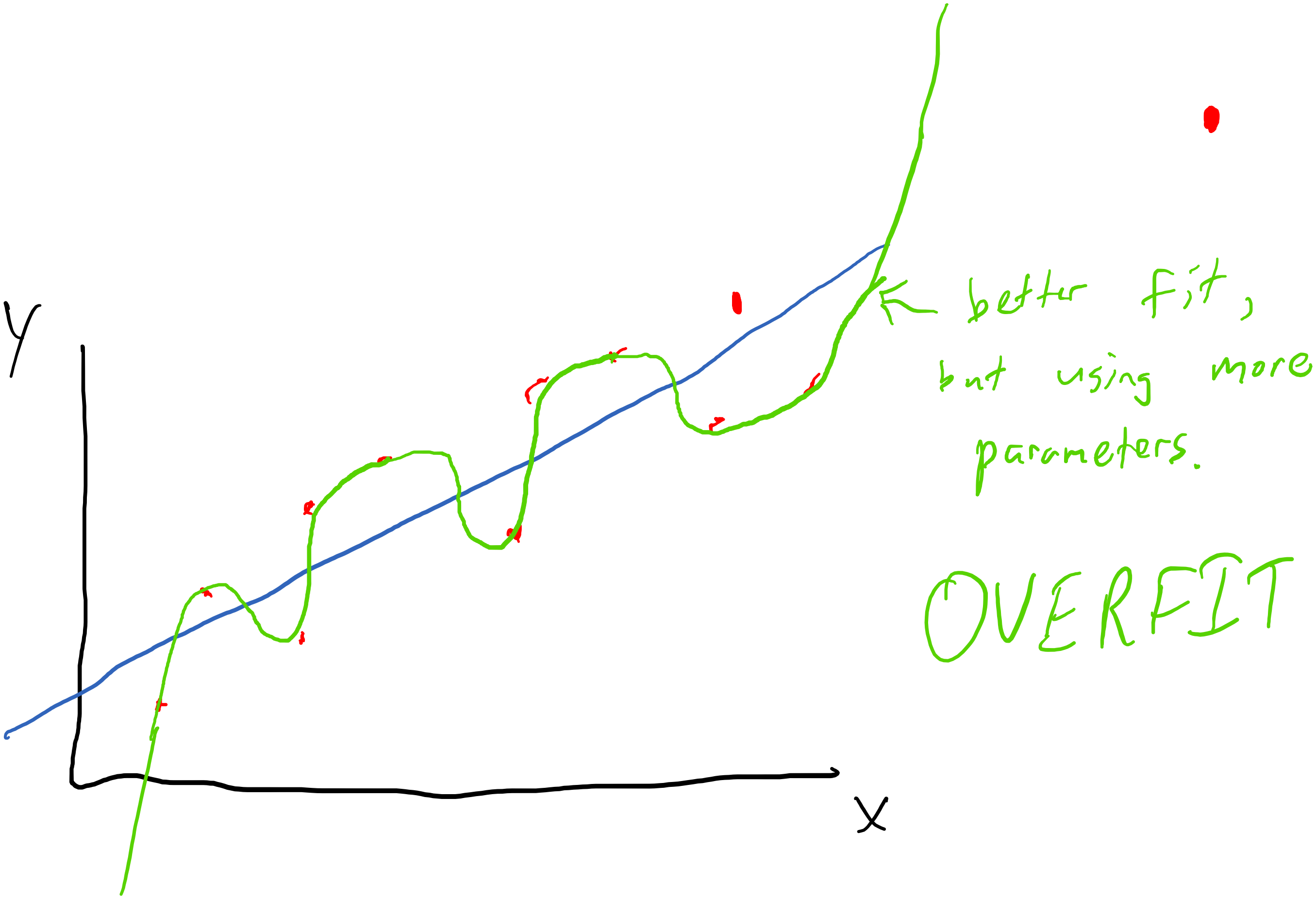
Recall: polynomial regression

- Given a collection of points, can approximate it with a polynomial function.



Be careful about too many parameters

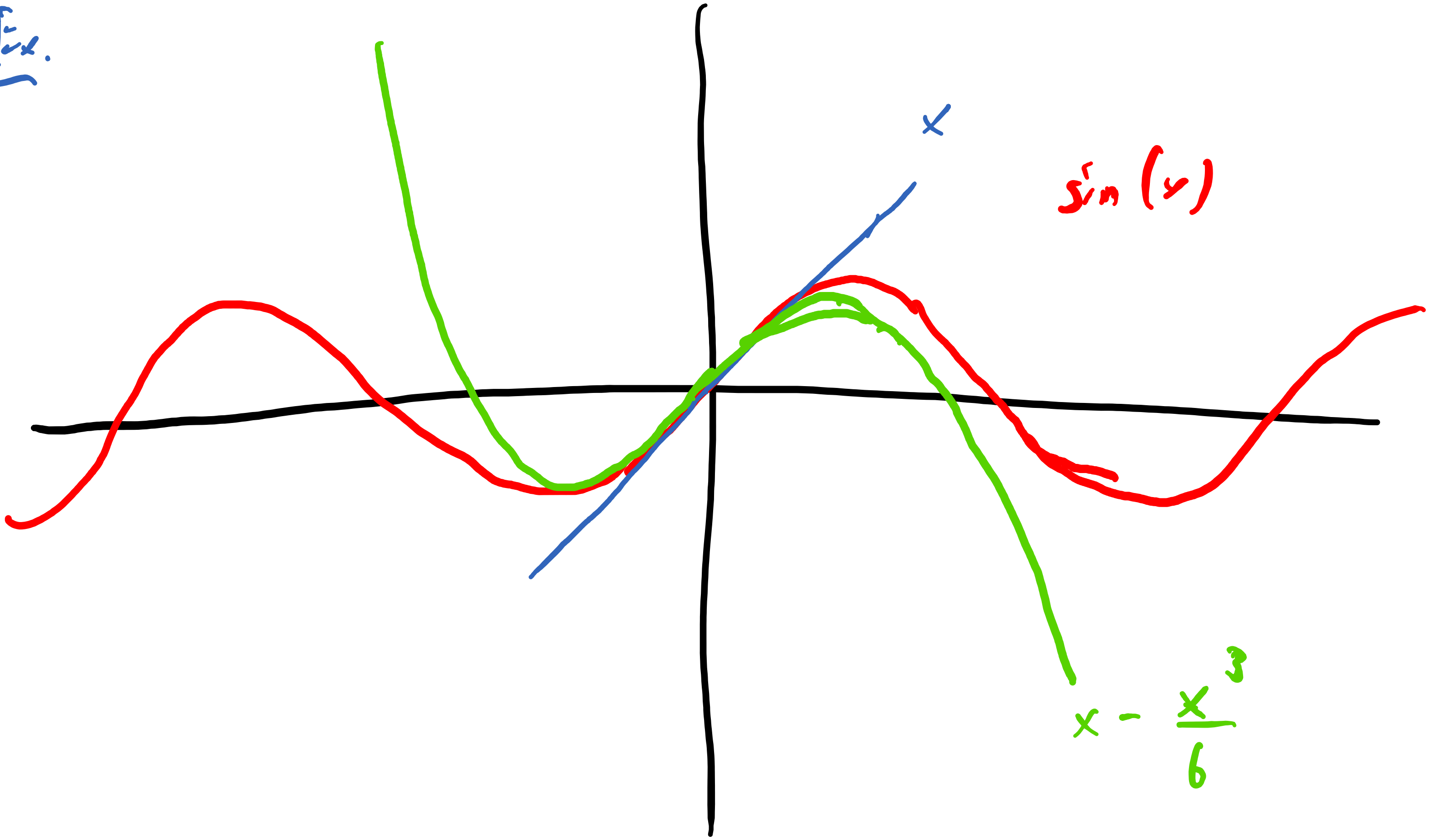
- The more parameters you have (e.g. in a polynomial regression), the better your mean squared error will be.
- However, sometimes, you will overfit to the data.
- John von Neumann: “with four parameters, I can fit an elephant, and with five I can make him wiggle his trunk”.



Approximating non-polynomial functions

- Sometimes, another "nice" function looks almost like a polynomial, at least locally.

Ex.



Formal power series

- A polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

can also be written as

$$p(x) = \sum_{i=0}^{n} a_i x^i$$

Ex. $p(x) = 1 + x + x^2 + x^3$

$$= \sum_{i=0}^{3} x^i$$

- A formal power series is the infinite sum where $n \to \infty$

$$p(x) = \sum_{i=0}^{\infty} a_i x^i$$

Ex. $p(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots \cdots = \sum_{i=0}^{\infty} x^i$

Convergence

- A formal power series $p(x) = \sum_{i=0}^{\infty} a_i x^i$ converges at a value x if the infinite sequence $x_0, x_1, x_2, \ldots,$ where $x_n = \sum_{i=0}^{n} a_i x^i$ converges to a limit as $n \to \infty$. It is divergent otherwise.

Ex. $p(x) = \sum_{i=0}^{\infty} x^i = 1 + x + x^2 + x^3 + x^4 + \ldots$

$p(\frac{1}{2}) = 1 + \frac{1}{2} + (\frac{1}{2})^2 + (\frac{1}{2})^3 + \ldots = 2$

$p(1) = 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + \ldots$ divergent

A: $p(0.5) = 1$
B: $p(0.5) = 2$
C: $p(0.5) = 3$
D: $p(0.5)$ is divergent
E: None of the above

If $|x| < 1$, $p(x) = \frac{1}{1-x}$

If $|x| \geq 1$, $p(x)$ is divergent

A: $p(1) = 1$
B: $p(1) = 2$
C: $p(1) = 3$
D: $p(1)$ is divergent
E: None of the above

Power series for a function

- A formal power series $p(x) = \sum_{i=0}^{\infty} a_i x^i$ can be thought of as a function whose domain is the interval of convergence.

Ex. $p(x) = 1 + x + x^2 + x^3 + x^4 + \dots$ is a function in the domain $(-1, 1)$

- Sometimes, we can express another function as a power series, at least on some interval of convergence.

Ex. $x^2 + 2x + 1$ is a poly and already a power series

$x^2 + 2x + 1 = \sum_{i=0}^{\infty} a_i \cdot x^i$ where $a_0 = 1$, $a_1 = 2$, $a_2 = 1$, $a_3 = 0$, $a_4 = 0, \dots$

Ex. $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots$ on the interval $(-1, 1)$

Manipulating power series

- How do we come up with a power series for a function?

Ex. $\underbrace{\frac{1}{1-x} = 1 + x + x^2 + \cdots}$ for $|x| < 1$

- Sometimes, we can manipulate it algebraically.

Check: $1 = (1-x) + x(1-x) + x^2(1-x) + x^3(1-x) + \cdots$

$1 = (1-x) + (x-x^2) + (x^2-x^3) + (x^3-x^4) + \cdots$ ✓

Ex. $p(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$

$p(x^3) = \frac{1}{1-x^3} = 1 + x^3 + x^6 + x^9 + x^{12} + \cdots$

Try it out

- $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \underline{\sum_{i=0}^{\infty} x^i}$
- What is $\frac{1}{1-2x}$? $= 1 + (2x) + (2x)^2 + (2x)^3 + (2x)^4 + \ldots$

$= 1 + 2x + 4x^2 + 8x^3 + 16x^4 + \ldots$

$= \sum_{i=0}^{\infty} (2x)^i = \sum_{i=0}^{\infty} 2^i x^i$

A: $\sum_{i=0}^{\infty} 2x^i$
B: $\sum_{i=0}^{\infty} 2^i x^i$
C: $\sum_{i=0}^{\infty} \frac{2^i x^i}{i!}$
D: $\sum_{i=0}^{\infty} \frac{x^{2i}}{i!}$
E: None of the above

- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{i=0}^{\infty} \frac{x^i}{i!}$
- What is e^{x^2}?

$= 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \ldots = \sum_{i=0}^{\infty} \frac{(x^2)^i}{i!} = \sum_{i=0}^{\infty} \frac{x^{2i}}{i!}$

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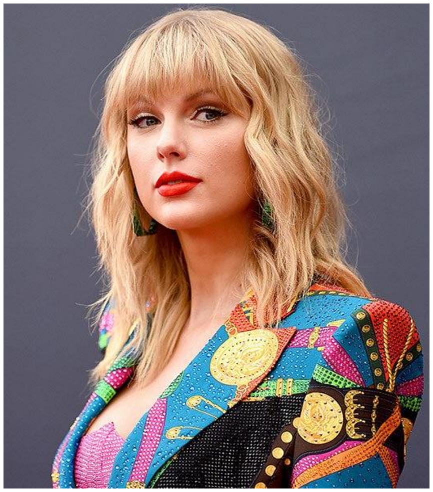

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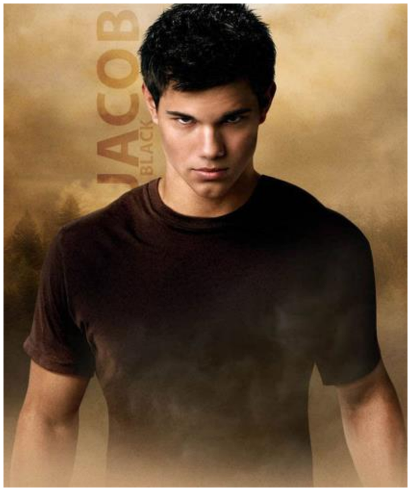

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Taylor series intuition

- If the space shuttle is moving at 10 m/s away from Earth, how far away from Earth is it after 1 minute?
- What if its speed is not constant?
- If the space shuttle is moving at 10 m/s, and it is constantly accelerating at 1 m/s^2, how far away is it after 1 minute?

A: 10 meters
B: 600 meters
C: 1000 meters
D: 2400 meters
E: None of the above

$\int \ddot{x} = \int 1 \rightarrow \dot{x}(t) = t + C$

$\dot{x}(0) = 10$

$x(0) = 0$

$\dot{x}(0) = 10 \Rightarrow C = 10$

$\int \dot{x}(t) = \int t + 10$

$x(t) = \frac{1}{2} t^2 + 10t + C_2$

$0 = x(0) = C_2 \Rightarrow x(t) = \frac{1}{2} t^2 + 10t$

$x(60) = 1800 + 600 = 2400$

- What if its acceleration is not constant?

Taylor series intuition (part 2)

- If we know all the derivatives of a polynomial at a point (e.g. at x=0), then we can reconstruct the polynomial.

Ex.

$$p(x) = 4 + 3x + 2x^2 + x^3$$
$$p'(x) = 3 + 4x + 3x^3$$
$$p''(x) = 4 + 6x$$
$$p'''(x) = 6$$
$$p''''(x) = 0$$
$$\vdots$$
$$p^{(n)}(x) = 0$$

$$p(0) = 4$$
$$p'(0) = 3$$
$$p''(0) = 4$$
$$p'''(0) = 6$$
$$p''''(0) = 0$$
$$\vdots$$
$$p^{(n)}(0) = 0$$

order of poly is 3

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$
$$p'(x) = a_1 + 2a_2 x + 3a_3 x^2$$
$$p''(x) = 2a_2 + 6a_3 x$$
$$p'''(x) = 6a_3$$

$$\begin{cases} p(0) = a_0 \\ p'(0) = a_1 \\ p''(0) = 2a_2 \\ p'''(0) = 6a_3 \end{cases}$$

$$\Rightarrow \quad a_0 = 4, \quad a_1 = 3, \quad a_2 = 2, \quad a_3 = 1$$

Taylor and Maclaurin series definitions

- The Maclaurin series of a function $f(x)$ is given by
$$\sum_{i=0}^{\infty}\frac{f^{(i)}(0)}{i!}x^i = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

- The Taylor series of a function $f(x)$ at a real number a is the power series
$$\sum_{i=0}^{\infty}\frac{f^{(i)}(a)}{i!}(x-a)^i$$
- The Maclaurin series is just the Taylor series at $a = 0$, and is the power series with matching derivatives at 0 with the original function $f(x)$.

Asides

- The Taylor series for any polynomial is the polynomial itself.
- A Taylor series may not necessarily converge at a point even if the function is well defined.
- A function may differ from the sum of its Taylor series, even if the Taylor series is convergent.
- However, for many common functions, the function and the sum of its Taylor series are equal in some radius of convergence.

e^x, $\sin x$, $\cos x$

Examples

- $f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{i=0}^{\infty} \frac{x^i}{i!}$

Note: $\frac{d}{dx} e^x = e^x$ so $f^{(i)}(0) = 1$

Thus the Maclaurin series is

$$\sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} x^i = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

Example

- What is the Maclaurin series for $f(x) = \cos x$?

$$\sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} x^i = 1 + \frac{0}{1!}x + \frac{-1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots$$

$$= \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i}}{(2i)!}$$

$f(x) = \cos x$ $\quad f(0) = 1$

$f'(x) = -\sin x$ $\quad f'(0) = 0$

$f''(x) = -\cos x$ $\quad f''(0) = -1$

$f'''(x) = \sin x$ $\quad f'''(0) = 0$

$\vdots$ $\quad \vdots$

Try it out

- What is the Maclaurin series for $f(x) = \sin x$?

$$\sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} x^i = 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + 0 - \frac{x^7}{7!} + \dots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$f(x) = \sin x \qquad f(0) = 0$

$f'(x) = \cos x \qquad f'(0) = 1$

$f''(x) = -\sin x \qquad f''(0) = 0$

$f'''(x) = -\cos x \qquad f'''(0) = -1$

$f''''(x) = \sin x \qquad \vdots$

A: $1 + x + x^2 + x^3 + x^4 + \cdots$

B: $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

C: $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$

D: $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$

E: None of the above

Proof of Euler's Equation

- $e^{ix} = \cos x + i \sin x$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \cdots$$

$$= 1 + ix - \frac{x^2}{2!} - i \cdot \frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} + \cdots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$= \cos x + i \sin x$$