Approximating functions
Lecture 11c: 2023-03-30

MAT A35 – Winter 2023 – UTSC

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Approximating $\sin x$

- Recall the Taylor series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

- Can you approximate $\sin 0.5$ by hand?

$$\sin 0.5 = 0.5 - \frac{0.5^3}{3!} + \frac{0.5^5}{5!} - \cdots$$

$$= 0.5 - \frac{0.125}{6} + \frac{0.03125}{120} - \cdots$$

$$= 0.5 - 0.02083$$

$$\approx 0.47916\cdots$$

$\sin 0.5 = 0.4794255$

- We can often approximate a function by cutting off its Taylor series after some number of terms.

Try it out

- What is $e^{0.5}$, approximated using the first 3 terms in its Maclaurin series?

$$e^x = \underbrace{1 + x + \frac{x^2}{2!}}_{} + \frac{x^3}{3!} + \cdots$$

$$e^{0.5} = 1 + 0.5 + \frac{0.5^2}{2!} = 1 + 0.5 + \frac{0.5^2}{2} = 1.5 + \frac{1}{8}$$

$$= 1.5 + 0.0625 = 1.5625$$

$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \dots \cdot n$

$2! = 1 \cdot 2 = 2$

$3! = 1 \cdot 2 \cdot 3 = 6$

$4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$

$\vdots$

A: 1

B: 1.5

C: 1.5625

D: 1.6487

E: None of the above

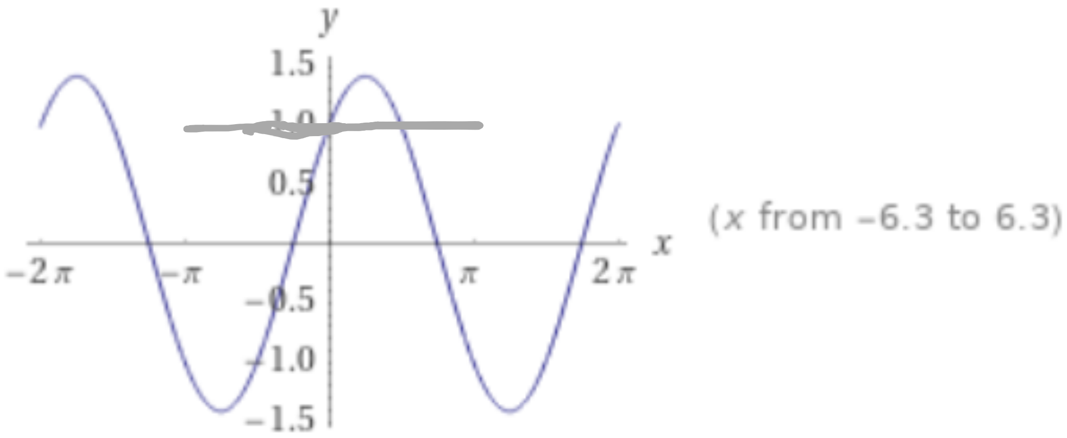
Constant approximation of a function

- If we keep just the first term of a Taylor series, we get a constant approximation.

Ex. $f(x) = \sin x + \cos x$

Taylor series $\underbrace{f(0)} + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$

$\sin x + \cos x \approx f(0) = 1$ around 0



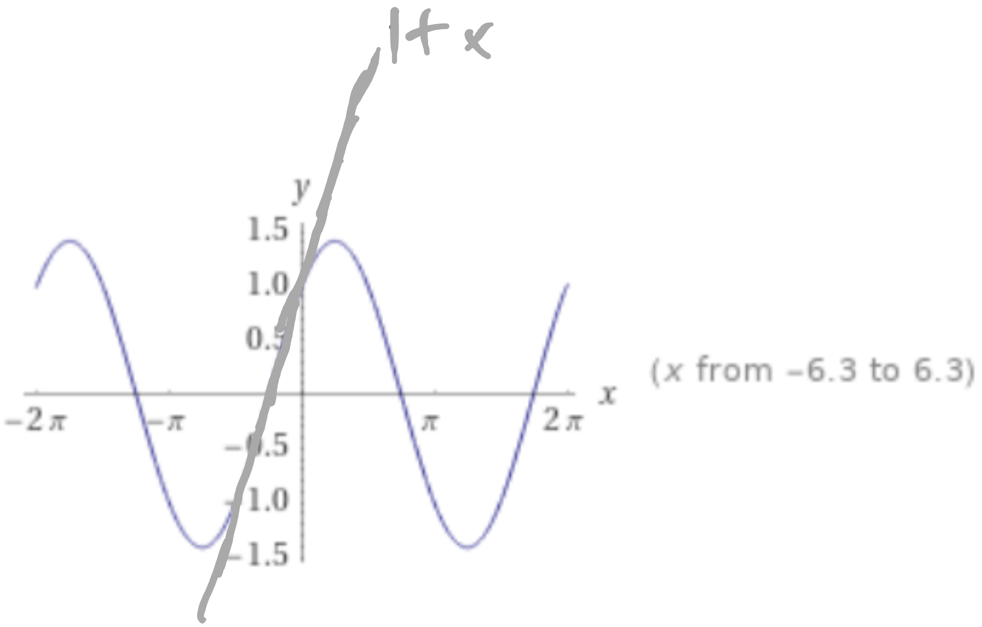
Linear approximation of a function

- If we keep just the first two terms of a Taylor series, we get a linear approximation.

Ex. $f(x) = \sin x + \cos x$ $\qquad f(0) = 1$

$f'(x) = \cos x - \sin x$ $\qquad f'(0) = 1$

Taylor series of $\sin x + \cos x \approx 1 + 1 \cdot x = 1 + x$



Quadratic approximation of a function

- If we keep just the first three terms of a Taylor series, we get a quadratic approximation.

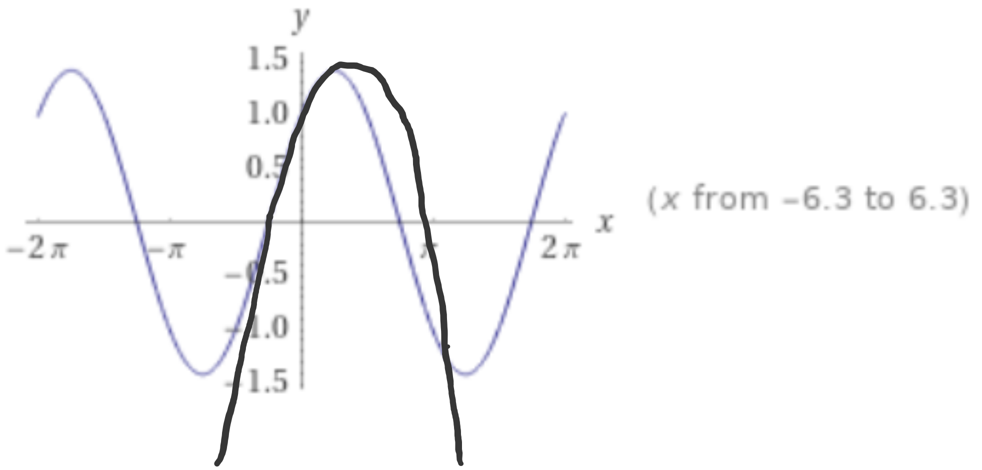
Ex. $f(x) = \sin x + \cos x$ $\qquad f(0) = 1$

$f'(x) = \cos x - \sin x$ $\qquad f'(0) = 1$

$f''(x) = -\sin x - \cos x$ $\qquad f''(0) = -1$

Taylor series $\quad f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots$

$$\sin x + \cos x \approx \underline{1 + x - \frac{x^2}{2}}$$



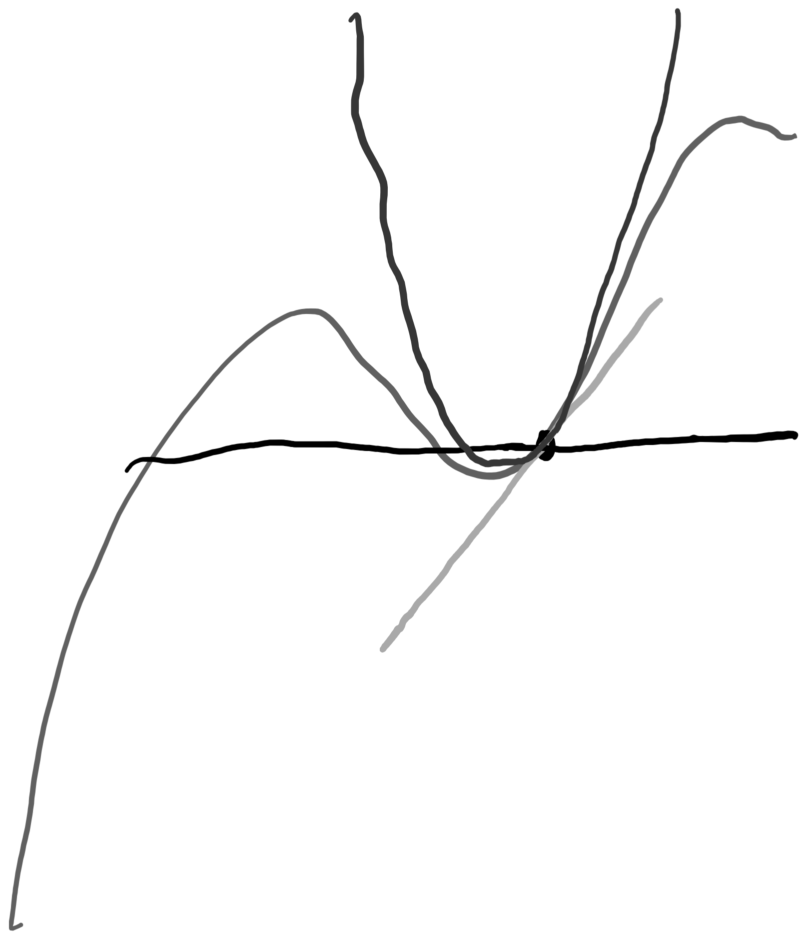
Qualitative analysis using approximation

cubic = 1st 4 terms of Taylor series

quartic = 1st 5 terms

⋮

- Often, the qualitative facts we are interested in can be captured by linear or quadratic approximation.
- Constant approximation: roughly where is the function?
- Linear approximation: is the function increasing or decreasing?
- Quadratic approximation: is the function concave up or down?



Multivariable functions

- A function $f(x)$ has a linear approximation at a of

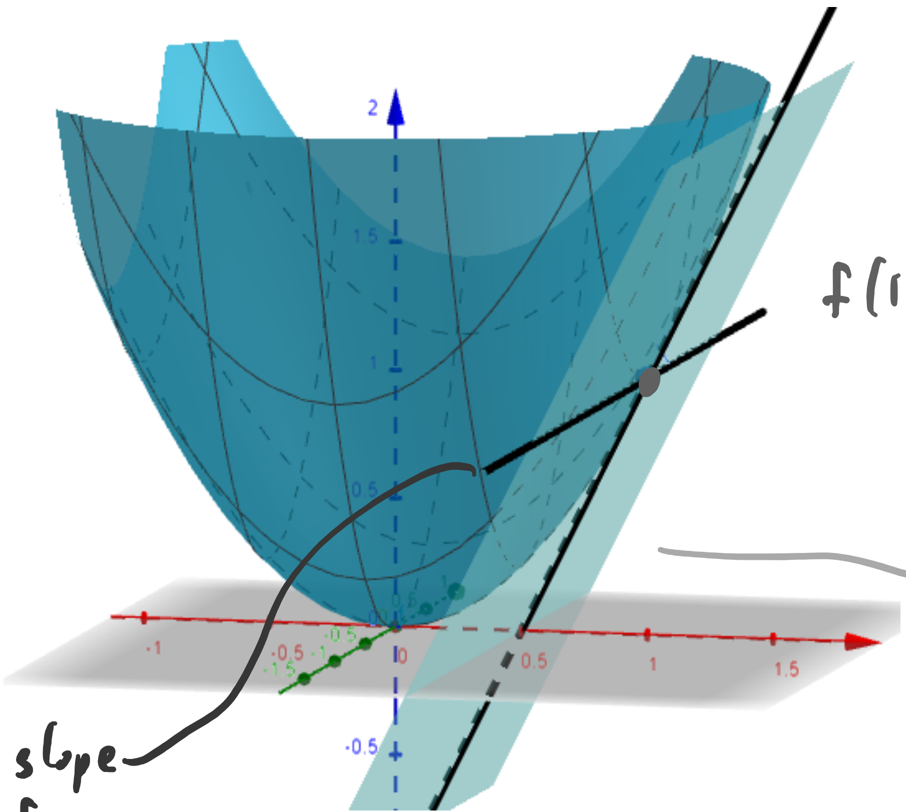
$$\underbrace{f(a)}_{\text{constant}} + \underbrace{f'(a)}_{\text{slope}}(x-a)$$

- A multivariable function $f(x,y)$ has a linear approximation at point (a,b) of

$$\underbrace{f(a,b)}_{\text{constant}} + \underbrace{\frac{\partial f(a,b)}{\partial x}}_{\substack{\text{slope in} \\ x\text{-direction}}}(x-a) + \underbrace{\frac{\partial f(a,b)}{\partial y}}_{\substack{\text{slope in} \\ y\text{-direction}}}(y-b)$$

$$f(x,y) = x^2 + y^2$$

slope $\frac{\partial f}{\partial x}(1,0) = 2$



$f(1,0) = 1$
constant approximation

plane is linear approximation

$f(x,y) = 1 + 2(x-1) + 0\cdot(y)$
$= -1 + 2x$

slope $\frac{\partial f}{\partial y}(1,0) = 0$

https://www.geogebra.org/3d/j8ntyjzw

Function $h: \mathbb{R}^2 \to \mathbb{R}^2$

- A multivariable function $f(x, y)$ has a linear approximation at point (a, b) of

$$f(a,b) + \frac{\partial f(a,b)}{\partial x}(x-a) + \frac{\partial f(a,b)}{\partial y}(y-b)$$

- So, $f(x,y) \approx f(a,b) + \begin{bmatrix} \frac{\partial f(a,b)}{\partial x} & \frac{\partial f(a,b)}{\partial y} \end{bmatrix} \begin{bmatrix} x-a \\ y-b \end{bmatrix}$
- Let $h(x,y) = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$, $h: \mathbb{R}^2 \to \mathbb{R}^2$
- Then

$$h(x,y) \approx \begin{bmatrix} f(a,b) \\ g(a,b) \end{bmatrix} + \overbrace{\begin{bmatrix} \frac{\partial f(a,b)}{\partial x} & \frac{\partial f(a,b)}{\partial y} \\ \frac{\partial g(a,b)}{\partial x} & \frac{\partial g(a,b)}{\partial y} \end{bmatrix}}^{\text{Jacobian matrix}} \begin{bmatrix} x-a \\ y-b \end{bmatrix}$$

Nonlinear phase portraits

- $h(x,y) \approx \begin{bmatrix} f(a,b) \\ g(a,b) \end{bmatrix} + \begin{bmatrix} \frac{\partial f(a,b)}{\partial x} & \frac{\partial f(a,b)}{\partial y} \\ \frac{\partial g(a,b)}{\partial x} & \frac{\partial g(a,b)}{\partial y} \end{bmatrix} \begin{bmatrix} x-a \\ y-b \end{bmatrix}$
- This approximation holds locally around every point, and therefore around each equilibrium.
- We can thus approximate its behavior by looking at the Jacobian matrix' eigenvalues.