Approximating functions
Lecture 11c: 2023-03-30

MAT A35 – Winter 2023 – UTSC

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Approximating $\sin x$

- Recall the Taylor series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

- Can you approximate $\sin 0.5$ by hand?

- We can often approximate a function by cutting off its Taylor series after some number of terms.

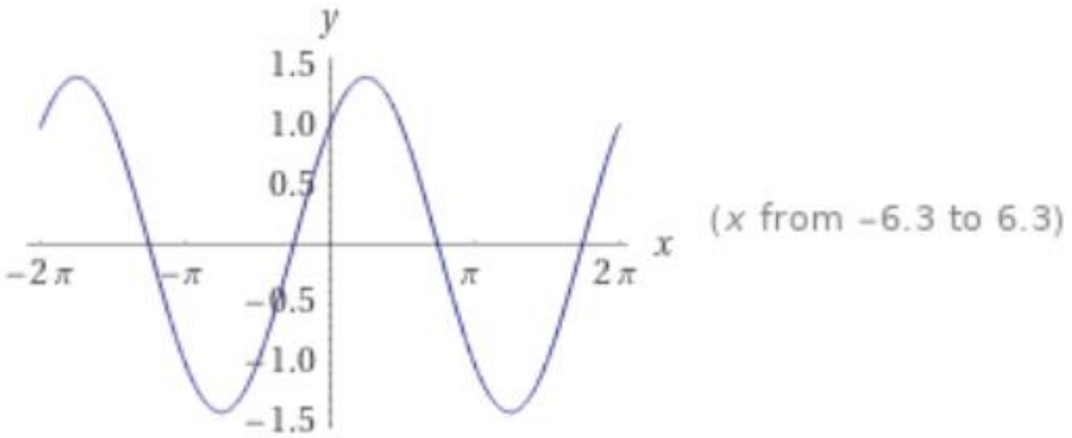
Try it out

- What is $e^{0.5}$, approximated using the first 3 terms in its Maclaurin series?

A: 1
B: 1.5
C: 1.5625
D: 1.6487
E: None of the above

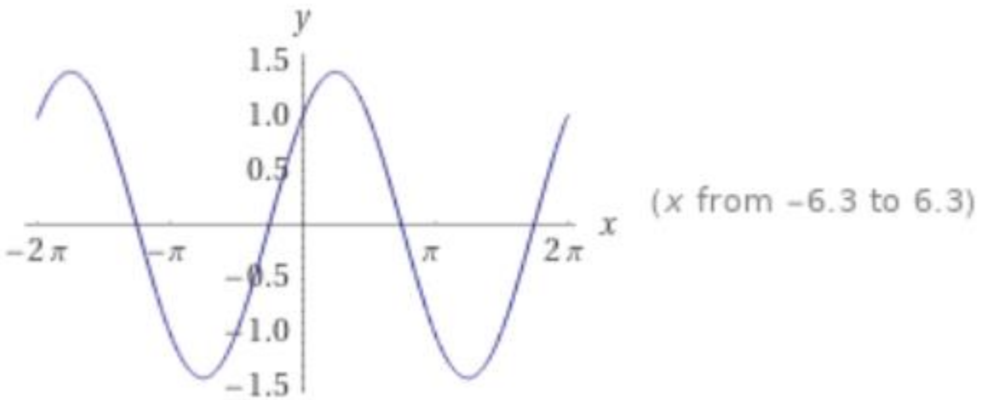
Constant approximation of a function

- If we keep just the first term of a Taylor series, we get a constant approximation.



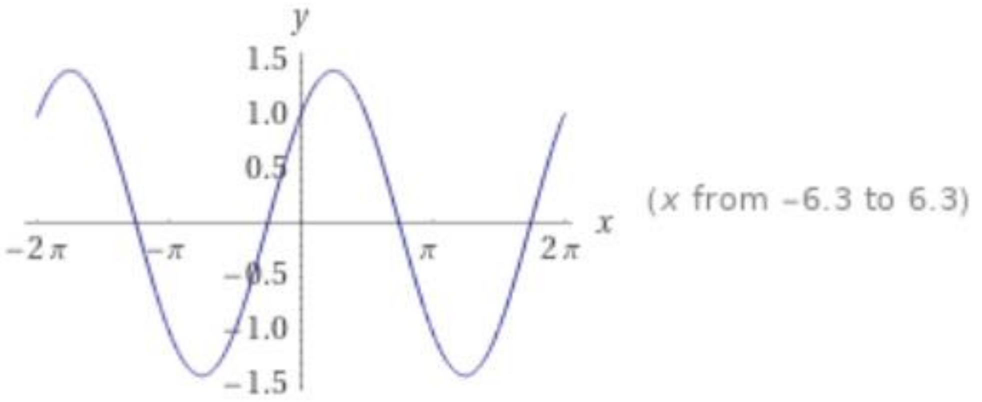
Linear approximation of a function

- If we keep just the first two terms of a Taylor series, we get a linear approximation.



Quadratic approximation of a function

- If we keep just the first three terms of a Taylor series, we get a quadratic approximation.



Qualitative analysis using approximation

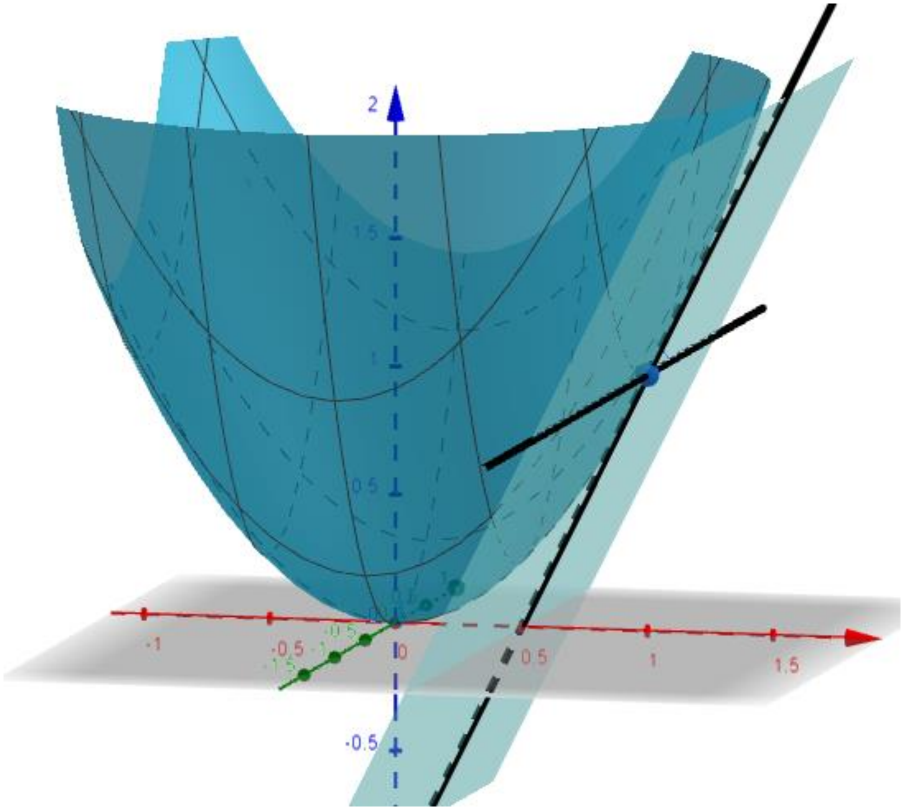
- Often, the qualitative facts we are interested in can be captured by linear or quadratic approximation.
- Constant approximation: roughly where is the function?
- Linear approximation: is the function increasing or decreasing?
- Quadratic approximation: is the function concave up or down?

Multivariable functions

- A function $f(x)$ has a linear approximation at a of
$$f(a) + f'(a)(x - a)$$

- A multivariable function $f(x, y)$ has a linear approximation at point (a, b) of
$$f(a, b) + \frac{\partial f(a, b)}{\partial x}(x - a) + \frac{\partial f(a, b)}{\partial y}(y - b)$$

$$f(x, y) = x^2 + y^2$$



https://www.geogebra.org/3d/j8ntyjzw

Function $h\colon \mathbb{R}^2 \to \mathbb{R}^2$

- A multivariable function $f(x, y)$ has a linear approximation at point (a, b) of
$$f(a,b) + \frac{\partial f(a,b)}{\partial x}(x-a) + \frac{\partial f(a,b)}{\partial y}(y-b)$$
- So, $f(x,y) \approx f(a,b) + \begin{bmatrix} \frac{\partial f(a,b)}{\partial x} & \frac{\partial f(a,b)}{\partial y} \end{bmatrix} \begin{bmatrix} x-a \\ y-b \end{bmatrix}$
- Let $h(x,y) = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$, $h\colon \mathbb{R}^2 \to \mathbb{R}^2$
- Then
$$h(x,y) \approx \begin{bmatrix} f(a,b) \\ g(a,b) \end{bmatrix} + \begin{bmatrix} \frac{\partial f(a,b)}{\partial x} & \frac{\partial f(a,b)}{\partial y} \\ \frac{\partial g(a,b)}{\partial x} & \frac{\partial g(a,b)}{\partial y} \end{bmatrix} \begin{bmatrix} x-a \\ y-b \end{bmatrix}$$

Nonlinear phase portraits

- $h(x,y) \approx \begin{bmatrix} f(a,b) \\ g(a,b) \end{bmatrix} + \begin{bmatrix} \frac{\partial f(a,b)}{\partial x} & \frac{\partial f(a,b)}{\partial y} \\ \frac{\partial g(a,b)}{\partial x} & \frac{\partial g(a,b)}{\partial y} \end{bmatrix} \begin{bmatrix} x-a \\ y-b \end{bmatrix}$
- This approximation holds locally around every point, and therefore around each equilibrium.
- We can thus approximate its behavior by looking at the Jacobian matrix' eigenvalues.