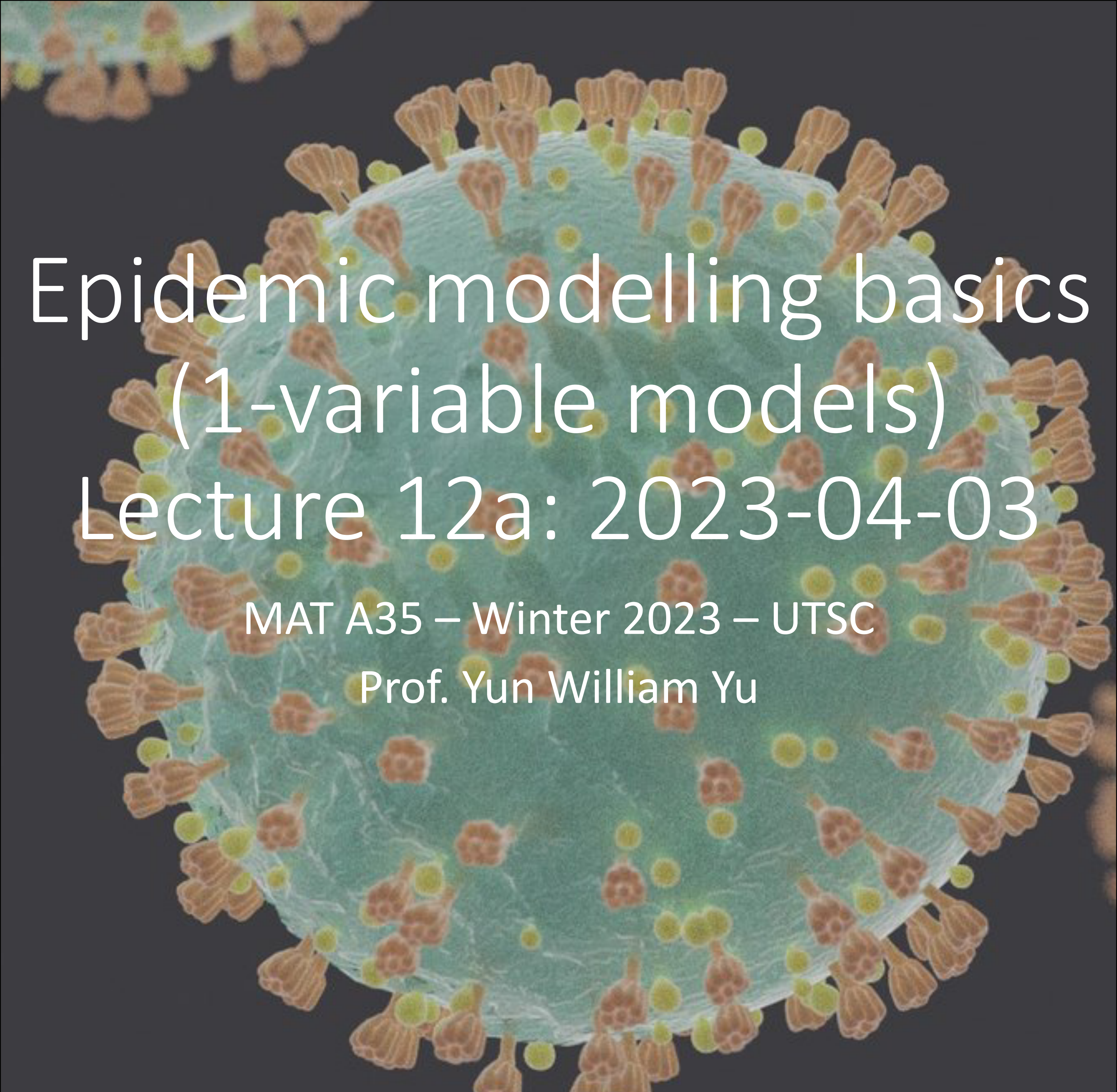


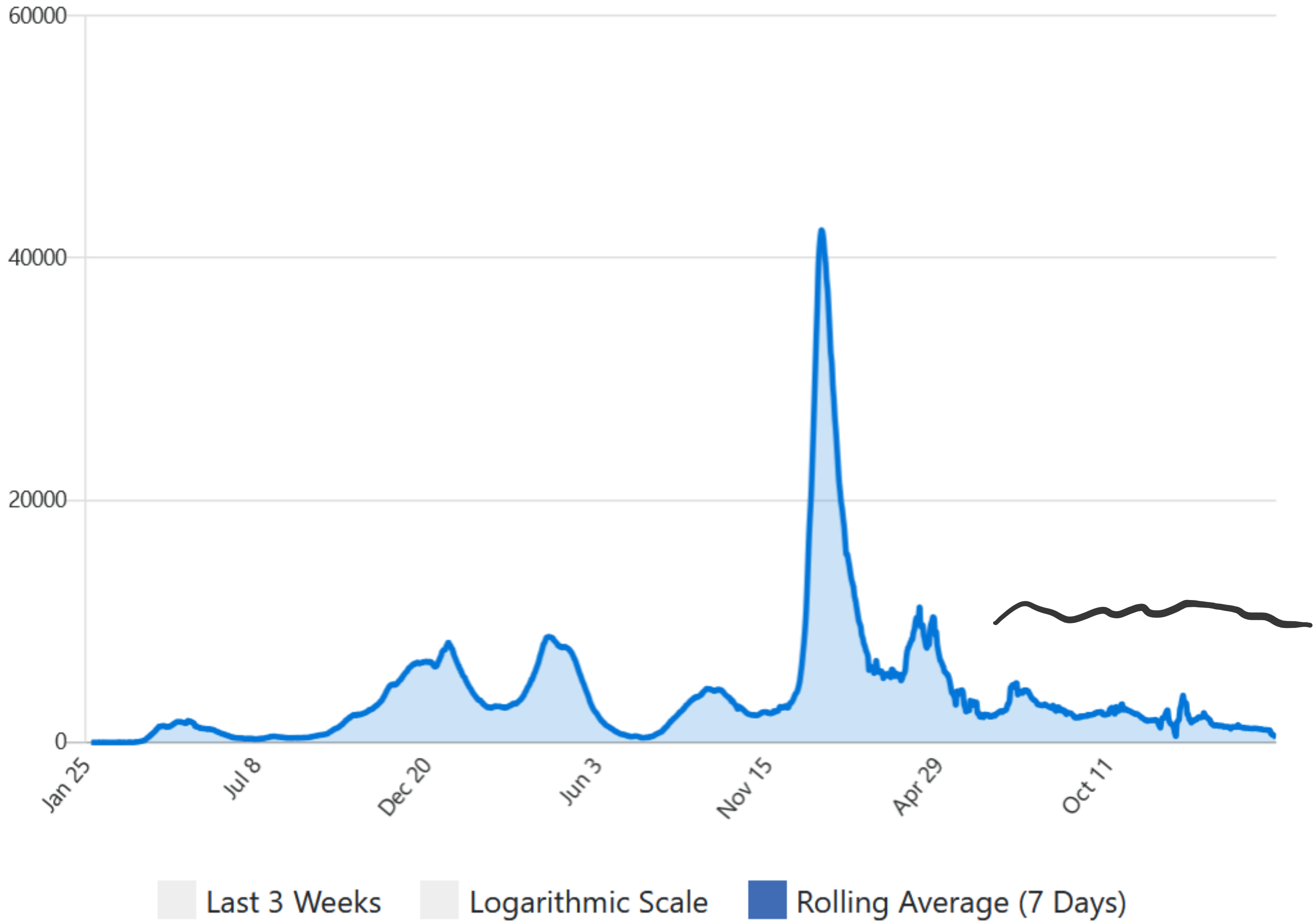
Epidemic modelling basics (1-variable models) Lecture 12a: 2023-04-03

MAT A35 – Winter 2023 – UTSC

Prof. Yun William Yu

Image credit: https://commons.wikimedia.org/wiki/File:Coronavirus_SARS-CoV-2.jpg

Epidemic curves – Covid19 in Canada



https://covid19tracker.ca/, 7-day rolling average, new cases

Infection rates

- Assumption 1: each infected individual infects other individuals at a constant positive rate β.

Model 1: Let $I(t)$ = # infected at time t

$$\frac{dI}{dt} = \beta I \Rightarrow \dot{I} = \beta I$$

$$\dot{I} - \beta I = 0$$

Char. eq: $\lambda - \beta = 0$

$$\Rightarrow \lambda = \beta$$

$$I(t) = c_1 e^{\beta t}$$

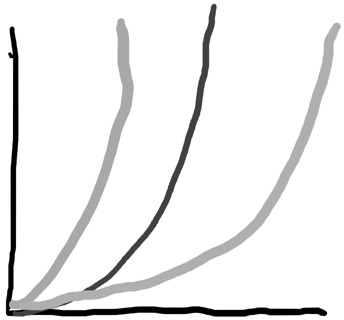
$$\Rightarrow I(0) = Ce^{\beta \cdot 0}$$

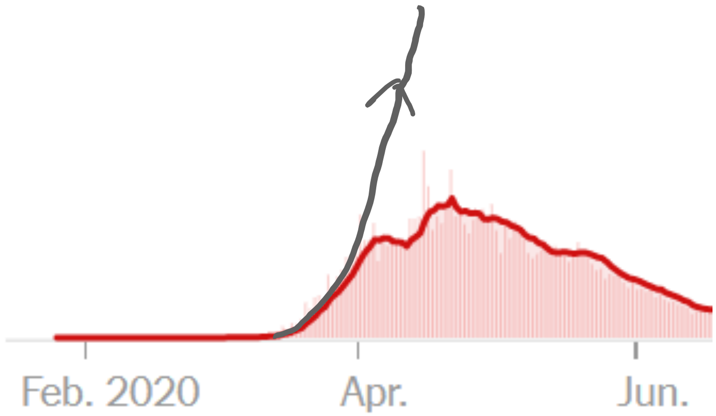
$$\Rightarrow C = I(0)$$

$$\Rightarrow I(t) = I(0)\, e^{\beta t}$$

Exponential model: $I(t) = I(0)e^{\beta t}$

- Let's focus on the initial months of the pandemic.
- Can use regression to find the good values for β, or even just trial and error.





- Model doesn't take into account finite population size.

What's wrong with the model?

A: Model is too simple

B: Cannot determine good β

C: Doesn't reflect the data

D: All of the above

E: None of the above

Compartmental models

- Assumption 2: there is a total fixed population size $N = I(t) + S(t)$, where S is the number of Susceptible individuals.

$$\left.\begin{aligned} \dot{I} &= \beta I \\ N &= I + S \end{aligned}\right\}$$

$$\begin{aligned} \dot{I} &= \beta I \\ S &= N - I \\ \dot{S} &= -\dot{I} = -\beta I \end{aligned}$$

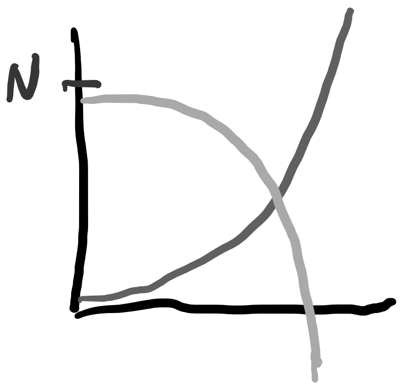
$$\boxed{S} \xrightarrow{\beta I} \boxed{I}$$

- Does this fix the problems from the previous slide?

$$\dot{I} = \beta I \quad \Rightarrow \quad I(t) = I(0)\, e^{\beta t}$$

$$S(t) = N - I(0)\, e^{\beta t}$$

A: Yes
B: No
C: Maybe
D: ???



Model still goes off to infinity infected, and even worse, negative susceptible

SI Model of Epidemics

- Modified assumption 1: The infection rate is proportional to the average number of times an infected individual encounters a susceptible individual in the population, assuming random encounters.

$$\beta \cdot I \cdot \frac{S}{N} = \text{infection rate}$$

$$N = S + I$$

$$\boxed{S} \xrightarrow{\frac{\beta S I}{N}} \boxed{I}$$

$$\dot{S} = -\frac{\beta S I}{N}$$

$$\dot{I} = \frac{\beta S I}{N}$$

Solving SI model qualitatively

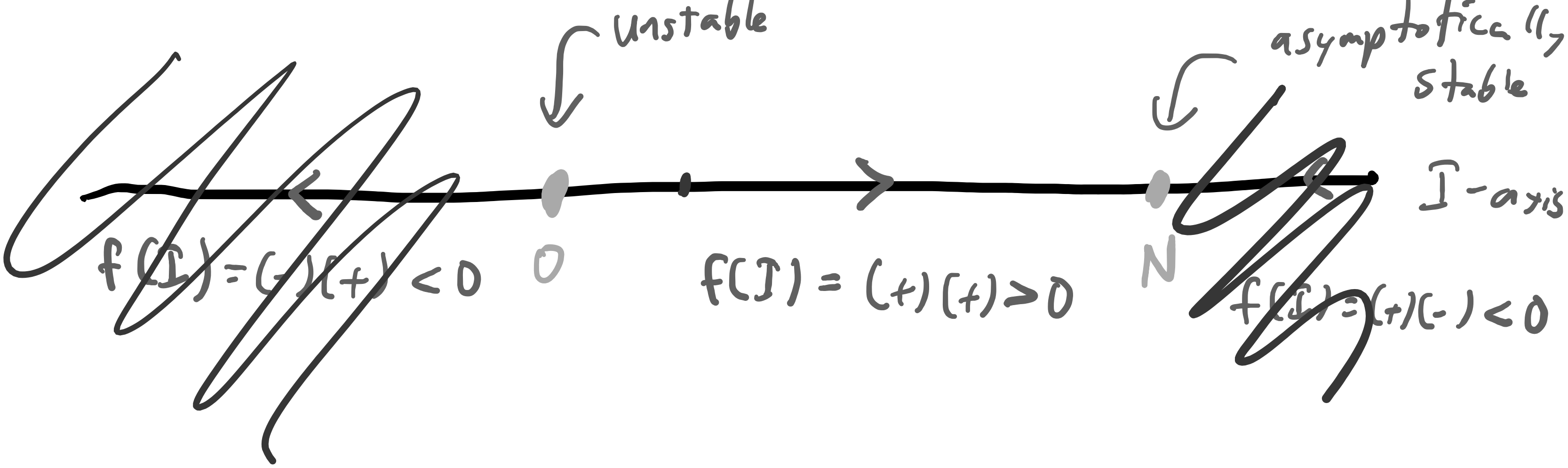
$$N = S(t) + I(t)$$
$$\dot{S} = -\frac{\beta S I}{N}$$
$$\dot{I} = \frac{\beta S I}{N}$$

$$S = N - I$$
$$\underline{\dot{I} = \frac{\beta I}{N}\cdot(N - I)} = f(I)$$

single-variable autonomous ODE

Equilibria:

$$0 = \dot{I} = \frac{\beta I}{N}(N - I)$$
$$\Rightarrow I = 0 \quad \text{or} \quad I = N$$



Solving the SI model exactly

- $\dot{I} = \frac{\beta I}{N}(N - I)$
- What methods should we use?

A: Separation of variables
B: Integrating factor
C: u-substitution
D: All of the above
E: None of the above

Integrating factor and u-substitution

- $\dot{I} = \frac{\beta I}{N}(N - I)$

$\dot{I} = \beta I - \frac{\beta I^2}{N}$

$\underbrace{\dot{I} - \beta I = -\frac{\beta I^2}{N}}$ Bernoulli ODE

Multiply by $-I^{-2}$

$\frac{dI}{dt} \cdot \frac{-1}{I^2} + \frac{\beta}{I} = \frac{\beta}{N}$

Let $u = \frac{1}{I}$, $du = \frac{-1}{I^2} dI$

$\underbrace{\frac{du}{dt} + \beta u = \frac{\beta}{N}}$ 1st order linear ODE

Char eq:

$\lambda + \beta = 0 \Rightarrow \lambda = -\beta$

$u_h = Ce^{-\beta t}$

Ansatz: $u_p = A$, $\dot{u}_p = 0$

$\Rightarrow \beta A = \frac{\beta}{N} \Rightarrow A = \frac{1}{N}$

$\Rightarrow u_p = \frac{1}{N}$

$\frac{1}{I} = u_{general} = u_h + u_p = Ce^{-\beta t} + \frac{1}{N}$

$$I = \frac{1}{Ce^{-\beta t} + \frac{1}{N}} = \frac{N}{1 + Ce^{-\beta t}}$$

Separation of variables

$$\dot{I} = \frac{\beta I}{N}(N - I) = \beta I \left(1 - \frac{I}{N}\right)$$

$$\frac{dI}{dt} = \beta I \left(1 - \frac{I}{N}\right)$$

separation of variables

$$\frac{dI}{I\left(1 - \frac{I}{N}\right)} = \beta\, dt$$

$$\frac{N}{I(N-I)}\, dI = \beta\, dt$$

partial fractions

$$\left[\frac{A}{I} + \frac{B}{N-I}\right] dI = \beta\, dt$$

$$\frac{A}{I} + \frac{B}{N-I} = \frac{N}{I(N-I)}$$

$$A(N-I) + BI = N$$

$$AN + I(B-A) = N$$

$$A = 1, \quad B - A = 0 \implies B = 1$$

$$\int \left[\frac{1}{I} + \frac{1}{N-I}\right] dI = \int \beta\, dt$$

$$-\ln|I| + \ln|N-I| = -\beta t + C$$

$$\ln\left|\frac{N-I}{I}\right| = -\beta t + C$$

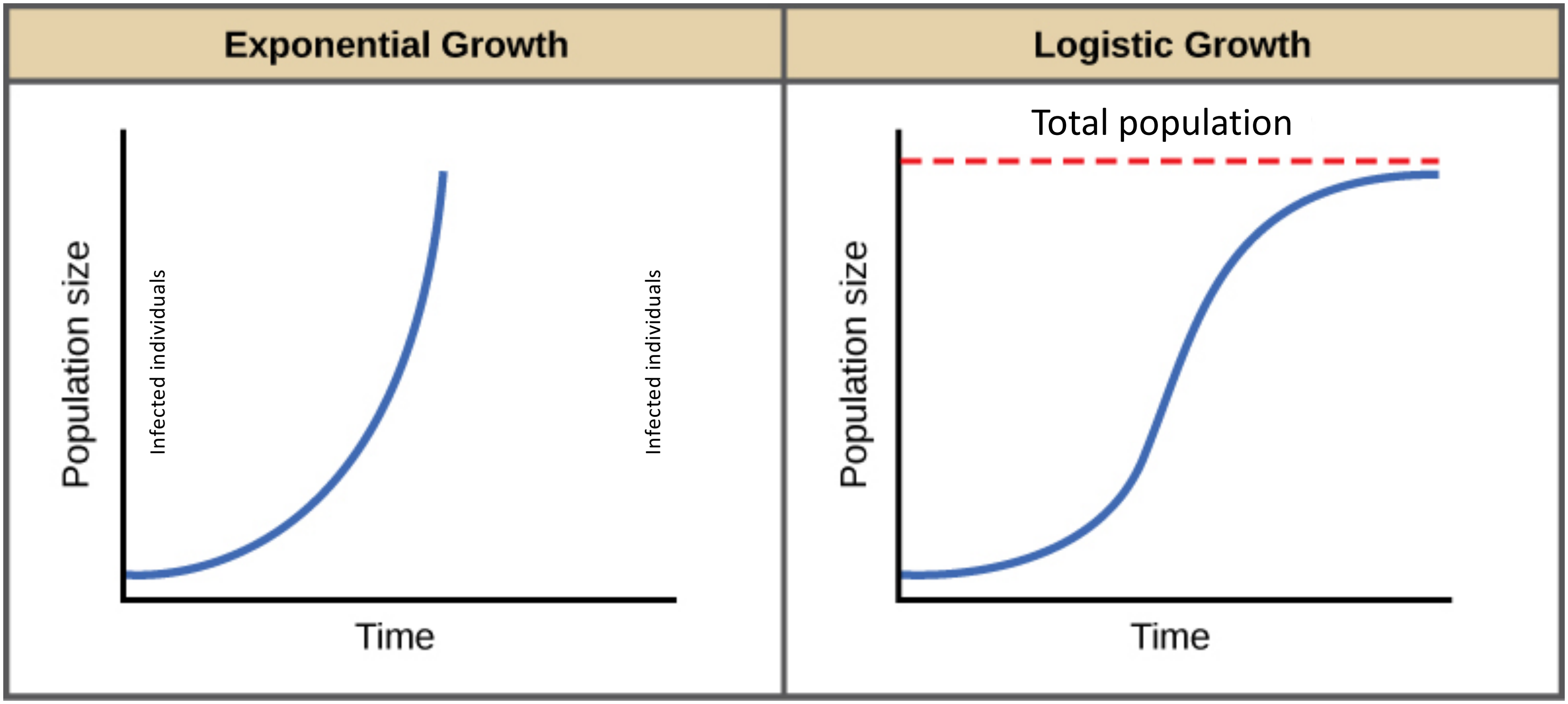
$$\frac{N-I}{I} = Ce^{-\beta t}$$

$$\frac{N}{I} = 1 + Ce^{-\beta t}$$

$$\implies I = \frac{N}{1 + Ce^{-\beta t}}$$

Logistic growth equation

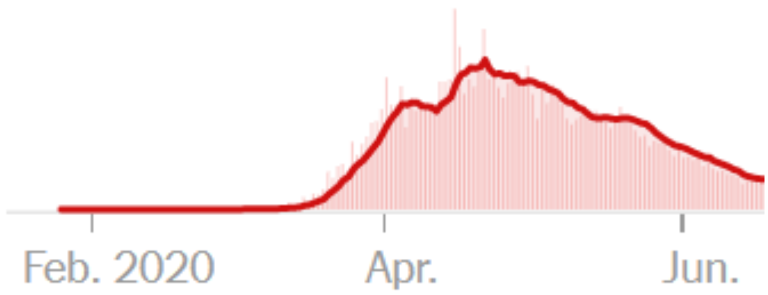
- SI model has logistic growth, which starts out like exponential growth, but levels out as everyone is infected.



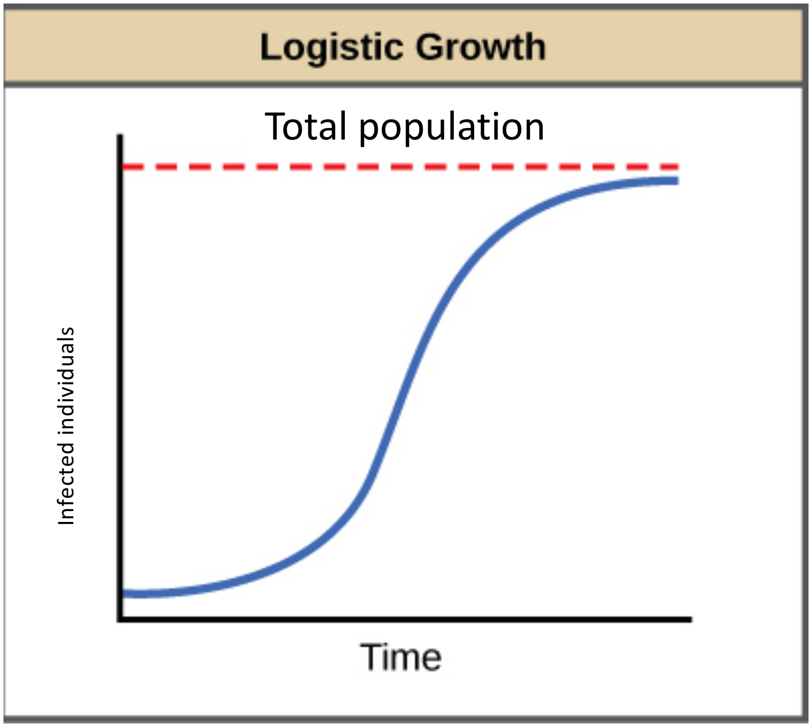
https://commons.wikimedia.org/wiki/File:Figure_45_03_01.jpg

Further improvements

- We now no longer go off to infinity, which is good.
- But, we still are missing the downward part of the epidemic curve.
- How do we get the number of infected to go back down in our model?



A: Add a recovery term
B: Add a mortality rate
C: Add an immunization rate
D: All of the above
E: None of the above



SIS Model

- Assumption 3: Individuals recover at rate γ, and become susceptible to re-infection.

$$S \underset{\gamma I}{\overset{\frac{\beta S I}{N}}{\rightleftarrows}} I$$

$$N = S(t) + I(t)$$

$$\left.\begin{aligned} \dot{S} &= \underline{\frac{-\beta}{N} S I} + \underline{\gamma I} \\ \dot{I} &= \underline{\frac{\beta}{N} S I} - \underline{\gamma I} \end{aligned}\right\} \quad \begin{aligned} S &= N - I \\ \dot{I} &= \frac{\beta}{N}(N - I) I - \gamma I \\ \dot{I} &= \underbrace{\left(\beta - \frac{\beta I}{N} - \gamma\right) I} \end{aligned}$$

Multiple cases

- $\frac{dI}{dt} = \left(\beta - \frac{\beta I}{N} - \gamma\right) I$
- Note, we have two parameters, β and γ, so there are three cases to consider.
- What are your guesses for behavior in each of the following?
- Case 1: $\beta = \gamma$ A
- Case 2: $\beta < \gamma$ A
- Case 3: $\beta > \gamma$ B

A: Disease dies out
B: Number of infected goes to nonzero constant
C: Number of infected oscillates up and down
D: All of the above
E: None of the above

Case 1: $\beta = \gamma$

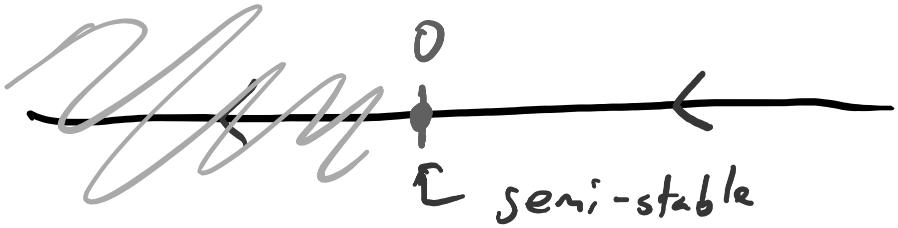
Infection rate = recovery rate

- $\dfrac{dI}{dt} = \left(\underline{\beta} - \dfrac{\beta I}{N} - \underline{\gamma}\right) I$

$\Rightarrow \dot{I} = -\dfrac{\beta I^2}{N}$

Eq. $\dot{I} = 0 \Rightarrow I = 0$

Note $-\dfrac{\beta I^2}{N} < 0 \quad \forall I \neq 0$



All positive values go to 0.

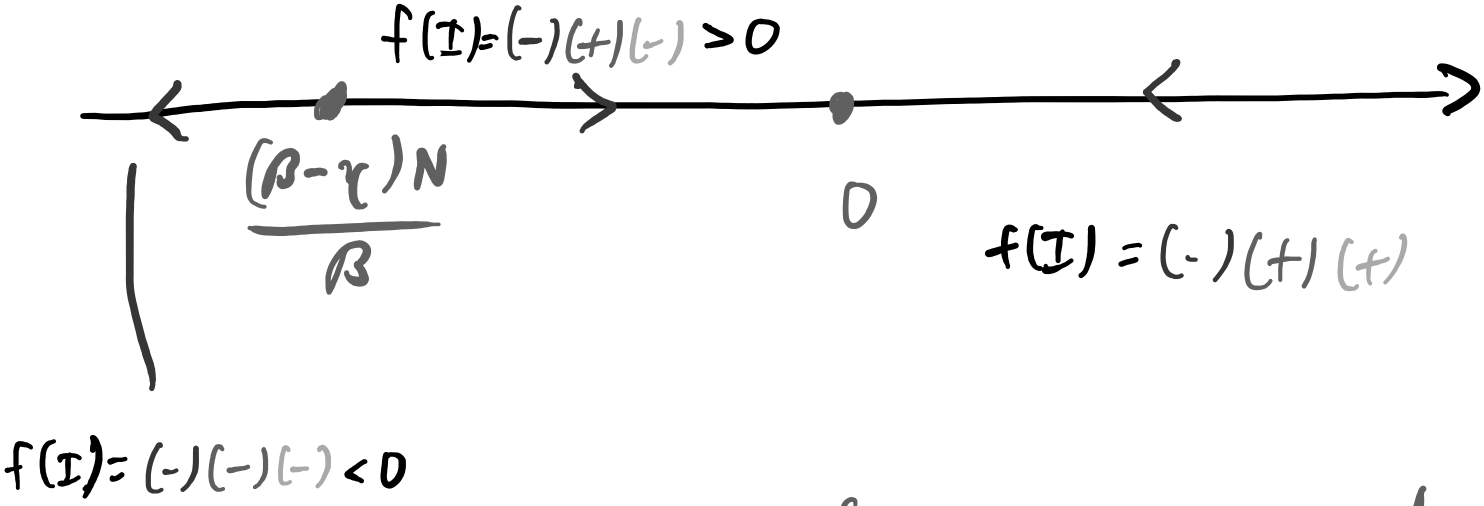
$\Rightarrow$ disease dies out

Case 2: $\beta < \gamma$

Infection rate < recovery rate

- $\frac{dI}{dt} = \left(\beta - \frac{\beta I}{N} - \gamma\right) I = \underline{(\beta-\gamma)}\left[1 - \frac{\beta}{(\beta-\gamma)N} I\right] I = f(I)$

Eq. $\dot{I} = 0 \Rightarrow I = 0$ or $I = \frac{(\beta-\gamma)N}{\beta} < 0$



irrelevant because negative

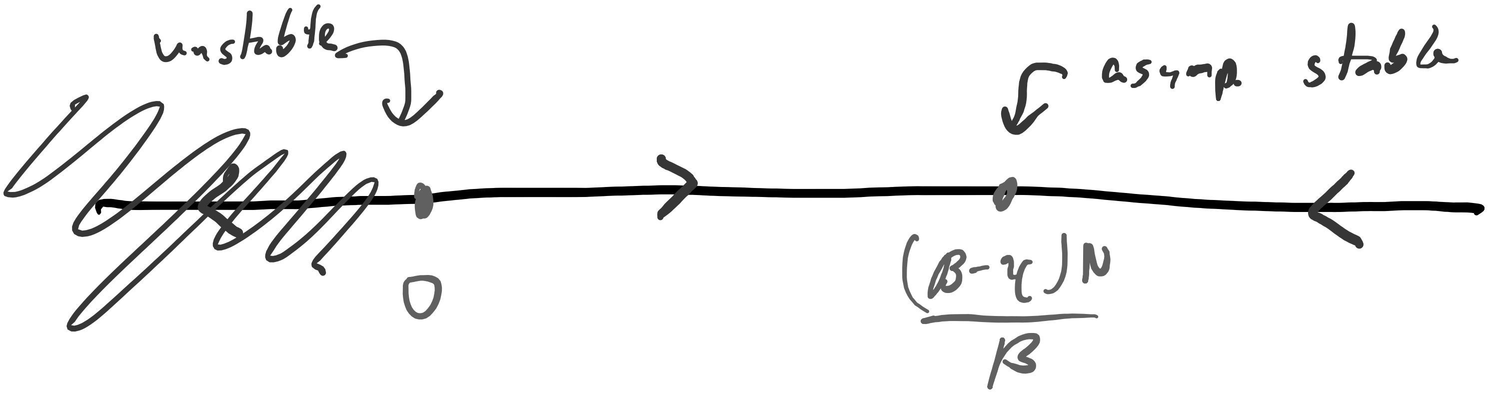
0 is the disease-free asymp. stable equilibrium

Case 3: $\beta > \gamma$

infection rate > recovery rate

- $\frac{dI}{dt} = \left(\beta - \frac{\beta I}{N} - \gamma\right) I = (\beta - \gamma)\left[1 - \frac{\beta}{(\beta-\gamma)N} \cdot I\right] I = f(I)$

Eq. $\dot{I} = 0 \Rightarrow I = 0$ or $I = \frac{(\beta - \gamma)}{\beta} \cdot N > 0$



Endemic asymp stable equilibrium at

$\frac{(\beta-\gamma)}{\beta}N$ infected.

Basic reproduction number

- Case 1: $\beta = \gamma$. Disease dies out.
- Case 2: $\beta < \gamma$. Disease dies out.
- Case 3: $\beta > \gamma$. Disease persists.
- In the SIS model, we call the ratio $R_0 = \frac{\beta}{\gamma}$ the *basic reproduction number* of the system because it is the number of secondary infections β caused during the infectious period $\frac{1}{\gamma}$.
- When $R_0 > 1$, disease persists.
- When $R_0 \leq 1$, disease dies out.